12

Stability Analysis

Ferenc Szidarovszky
University of Arizona

A. Terry Bahill
University of Arizona

12.1 Introduction

In this chapter, which is based on Szidarovszky and Bahill (1998), we discuss **stability**. We start by discussing interior stability, where the stability of the state trajectory or equilibrium state is examined and then we discuss exterior stability, in which we guarantee that a bounded input always evokes a bounded output. We present four techniques for examining **interior stability**: (1) Lyapunov functions, (2) checking the boundedness or limit of the fundamental matrix, (3) finding the location of the eigenvalues for state-space notation, and (4) finding the location of the poles in the complex frequency plane of the closed-loop transfer function. We present two techniques for examining **exterior** (or BIBO) *stability* (1) use of the weighting pattern of the system and (2) finding the location of the eigenvalues for state-space notation.

Proving stability with Lyapunov functions is very general: it even works for nonlinear and time-varying systems. It is also good for doing proofs. However, proving the stability of a system with Lyapunov functions is difficult. And failure to find a Lyapunov function that proves a system is stable does not prove that the system is unstable. The next technique we present, finding the fundamental matrix, requires the solution of systems of differential equations, or in the time invariant case, the computation of the eigenvalues. Determining the eigenvalues or the poles of the transfer function is sometimes difficult, because it requires factoring high-order polynomials. However, many commercial software packages are available for this task. We think most engineers would benefit by having one of these computer programs. Jamshidi et al. (1992) and advertisements in technical publications such as the *IEEE Control Systems Magazine* and *IEEE Spectrum* describe many appropriate software packages. The last concept we present, bounded-input, bounded-output stability, is very general.

Let us begin our discussion of stability and instability of systems informally. In an *unstable system,* the state can have large variations, and small inputs or small changes in the initial state may produce large variations in the output. A common example of an unstable system is illustrated by someone pointing the microphone of a public address (PA) system at a speaker; a loud high-pitched tone results. Often instabilities are caused by too much gain; so to quiet the PA system, decrease the gain by pointing the microphone away from the speaker. Discrete systems can also be unstable. A friend of ours once provided an example. She was sitting in a chair reading and she got cold. So she went over and turned up the thermostat on the heater. The house warmed up. She got hot, so she got up and turned down the thermostat. The house cooled off. She got cold and turned up the thermostat. This process continued until someone finally suggested that she put on a sweater (reducing the gain of her heat loss system). She did, and was much more comfortable. We called this a discrete system, because she seemed to sample the environment and produce outputs at discrete intervals about 15 minutes apart.

12.2 Using the State of the System to Determine Stability

The stability of a system can be defined with respect to a given equilibrium point in state space. If the initial state x_0 is selected at an equilibrium state $\bar{x}$ of the system, then the state will remain at $\bar{x}$ for all future time. When the initial state is selected close to an equilibrium state, the system might remain close to the equilibrium state or it might move away. In this section, we introduce conditions that guarantee that whenever the system starts near an equilibrium state, it remains near it, perhaps even converging to the equilibrium state as time increases. For simplicity, only time-invariant systems are considered in this section. Time-variant systems are discussed in Section 12.5.

Continuous, time-invariant systems have the form

$$\dot{x}(t) = f(x(t)) \tag{12.1}$$

and discrete, time-invariant systems are modeled by the difference equation

$$x(t+1) = f(x(t)) \tag{12.2}$$

Here we assume that $f : X \to R^n$, where $X \subset R^n$ is the state space. We also assume that function f is continuous; furthermore, for arbitrary initial state $x_0 \in X$, there is a unique solution of the corresponding initial value problem $x(t_0) = x_0$, and the entire trajectory $x(t)$ is in X. Assume furthermore that denotes the initial time period of the system. It is also known that the vector $\bar{x} \in X$ is an equilibrium state of the continuous system, Equation 12.1, if and only if $f(\bar{x}) = 0$, and it is an equilibrium state of the discrete system, Equation 12.2, if and only if $\bar{x} = f(\bar{x})$. In this chapter the equilibrium of a system will always mean the equilibrium *state*, if it is not specified otherwise. In analyzing the dependence of the state trajectory $x(t)$ on the selection of the initial state x_0 nearby the equilibrium, the following stability types are considered.

Definition 12.1

1. An equilibrium state $\bar{x}$ is *stable* if there is an $\varepsilon_0 > 0$ with the following property For all ε_1, $0 < \varepsilon_1 < \varepsilon_0$, there is an $\varepsilon > 0$ such that if $\|\bar{x} - x_0\| < \varepsilon$, then $\|\bar{x} - x(t)\| < \varepsilon_1$ for all $t > t_0$.
2. An equilibrium state $\bar{x}$ is *asymptotically stable* if it is stable and there is an $\varepsilon > 0$ such that whenever $\|\bar{x} - x_0\| < \varepsilon$, then $x(t) \to \bar{x}$ as $t \to \infty$.
3. An equilibrium state $\bar{x}$ is *globally asymptotically stable* if it is stable and with arbitrary initial state $x_0 \in X, x(t) \to \bar{x}$ as $t \to \infty$.

The first definition says an equilibrium state $\bar{x}$ is stable if the entire trajectory $x(t)$ is closer to the equilibrium state than any small ε_1, if the initial state x_0 is selected close enough to the equilibrium state. For asymptotic stability, in addition $x(t)$ converges to the equilibrium state as $t \to \infty$. If equilibrium state is globally asymptotically stable, then $x(t)$ converges to the equilibrium state regardless of how the initial state x_0 is selected.

These stability concepts are called *internal*, because they represent properties of the state of the system. They are illustrated in Figure 12.1, where the block dots are the initial states and $\bar{x}$ is the origin.

In the electrical engineering literature sometimes our stability definition is called marginal stability, and our asymptotic stability is called stability.

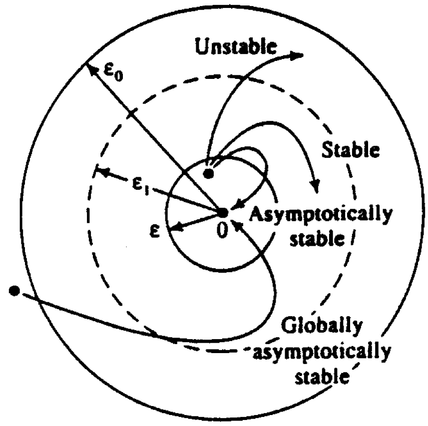

FIGURE 12.1 Stability concepts. (*Source*: F. Szidarovszky and A.T. Bahill, *Linear Systems Theory*, Boca Raton, Fla.: CRC Press, 1998, p. 199. With Permission.)

12.3 Lyapunov Stability Theory

Assume that $\bar{x}$ is an equilibrium state of a continuous or discrete system, and let Ω denote a subset of the state space X such that $\bar{x} \in \Omega$.

Definition 12.2

A real-valued function V defined on Ω is called a Lyapunov function, if

1. V is continuous;
2. V has a unique global minimum at $\bar{x}$ with respect to all other points in Ω;
3. For any state trajectory $x(t)$ contained in Ω, $V(x(t))$ is nonincreasing in t.

The Lyapunov function can be interpreted as the generalization of the energy function in electrical systems. The first requirement simply means that the graph of V has no breaks. The second requirement means that the graph of V has its lowest point at the equilibrium, and the third requirement generalizes the well-known fact of electrical systems, that the energy in a free electrical system with resistance always decreases, unless the system is at rest.

Theorem 12.1

Assume that there exists a Lyapunov function V on the spherical region

$$\Omega = \{x \, \| x - \bar{x} \| < \varepsilon_0\} \tag{12.3}$$

where $\varepsilon_0 > 0$ is given; furthermore $\Omega \subseteq X$. Then the equilibrium state is stable.

Theorem 12.2

Assume that in addition to the conditions of Theorem 12.1, the Lyapunov function $V(x(t))$ is strictly decreasing in t unless $x(t) = \bar{x}$. Then the equilibrium state is asymptotically stable.

Theorem 12.3

Assume that the Lyapunov function defined on the entire state space X, $V(x(t))$ is strictly decreasing in t unless $x(t) = \bar{x}$; furthermore, $V(x)$ tends to infinity as any component of $\boldsymbol{x}$ gets arbitrarily large in magnitude. Then the equilibrium state is globally asymptotically stable.

Example 12.1

Consider the differential equation

$$\dot{x} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which describes a harmonic oscillator.

The stability of the equilibrium state $(1/\omega,\ 0)^{\mathrm{T}}$ can be verified directly by using Theorem 12.1 without computing the solution. Select the Lyapunov function

$$V(x) = (x - \bar{x})^{\mathrm{T}}(x - \bar{x}) = \ \|x - \bar{x}\ \|_2^2$$

where the Euclidean norm is used.

This is continuous in x; furthermore, it has its minimal (zero) value at $x = \bar{x}$. Therefore, to establish the stability of the equilibrium state we have to show that $V(x(t))$ is decreasing. Simple differentiation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) = 2(x - \bar{x})^{\mathrm{T}}\, \dot{x} = 2(x - \bar{x})^{\mathrm{T}}(Ax + b)$$

with

$$A = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

That is, with $x = (x_1, x_2)^{\mathrm{T}}$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) &= 2\left(x_1 - \frac{1}{\omega}, x_2\right)\begin{pmatrix} \omega x_2 \\ -\omega x_1 + 1 \end{pmatrix} \\ &= 2\big(\omega x_1 x_2 - x_2 - \omega x_1 x_2 + x_2\big) = 0 \end{aligned}$$

Therefore, function $V(x(t))$ is a constant, which is a nonincreasing function. That is, all conditions of Theorem 12.1 are satisfied, which implies the stability of the equilibrium state.

Theorem 12.1 to Theorem 12.3 guarantee, respectively, the stability, asymptotic stability and global asymptotic stability of the equilibrium state, if a Lyapunov function is found. Failure to find such a Lyapunov function does not mean that the system is unstable or that the stability is not asymptotic or globally asymptotic. It only means that you were not clever enough to find a Lyapunov function that proved stability.

12.4 Stability of Time-Invariant Linear Systems

This section is divided into two subsections. In the first subsection, the stability of linear time invariant systems given in state-space notation is analyzed. In the second subsection, methods based on transfer functions are discussed.

Stability Analysis with State-Space Notation

Consider the time-invariant continuous linear system

$$\underline{\mathbf{x}} = \mathbf{Ax} + \mathbf{b} \tag{12.4}$$

and the time-invariant discrete linear system

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{b} \tag{12.5}$$

Assume that $\bar{x}$ is an equilibrium state, and let $\boldsymbol{\phi}(t, t_0)$ denote the fundamental matrix.

Theorem 12.4

1. The equilibrium state $\bar{x}$ is stable if and only if $\boldsymbol{\phi}(t, t_0)$ is bounded for $t \geqslant t_0$.
2. The equilibrium state $\bar{x}$ is asymptotically stable if and only if $\boldsymbol{\phi}(t, t_0)$ is bounded and tends to zero as $t \longrightarrow \infty$.

In the case of linear systems, asymptotic stability and global asymptotic stability are equivalent.

We use the symbol s to denote complex frequency, i.e., $s = \sigma + j\omega$. For specific values of s, such as eigenvalues and poles, we use the symbol λ.

Theorem 12.5

1. Assume that for all eigenvalues λ_i of $\boldsymbol{A}$, Re $\lambda_i \leqslant 0$ in the continuous case (or $|\lambda_i| \leqslant 1$ in the discrete case), and all eigenvalues with the property Re $\lambda_i = 0$ (or $|\lambda_i| = 1$) have single multiplicity; then the equilibrium state is stable.
2. The stability is asymptotic if and only if for all i, Re $\lambda_i < 0$ (or $|\lambda_i| < 1$).
3. If for at least one eigenvalue of $\boldsymbol{A}$, Re $\lambda_i > 0$(or $|\lambda_i| > 1$) then the equilibrium is unstable.

Remark 1. Note that Part 1 gives only sufficient conditions for the stability of the equilibrium state. As the following examples show, these conditions are not necessary.

If there is at least one multiple eigenvalue with zero-real part (unit absolute value) then we cannot decide the stability of the equilibrium based on only the eigenvalues. In such cases the boundedness of the fundamental matrix has to be checked.

Example 12.2

Consider first the continuous system $\dot{x} = Ox$,where $\mathbf{O}$ is the zero matrix. Note that all constant functions $x(t) \equiv \bar{x}$ are solutions and also equilibrium states. Since

$$\phi(t, t_0) = e^{\mathbf{O}(t-t_0)} = \mathbf{I}$$

is bounded (being independent of t), all equilibrium states are stable, but $\boldsymbol{O}$ has only one eigenvalue $\lambda_1 = 0$ with zero real part and multiplicity n, where n is the order of the system.

Consider next the discrete systems $x(t+1) = \mathbf{I}\mathbf{x}(t)$, when all constant functions $x(t) \equiv \bar{x}$ are also solutions and equilibrium states. Furthermore,

$$\phi(t, t_0) = \mathbf{A}^{t-t_0} = \mathbf{I}^{t-t_0} = \mathbf{I}$$

which is obviously bounded. Therefore, all equilibrium states are stable, but the condition of Part 1 of the theorem is violated again.

Remark 2. The following extension of Theorem 12.5 can be proven. The equilibrium state is stable if and only if for all eigenvalues of $\boldsymbol{A}$, Re $\lambda_i \leqslant 0$ (or $|\lambda_i| \leqslant 1$), and if λ_i is a repeated eigenvalue of $\boldsymbol{A}$ such that Re $\lambda_i = 0$ (or $|\lambda_i| = 1$), then the size of each block containing λ_i in the Jordan canonical form of $\boldsymbol{A}$ is 1×1.

Remark 3. The equilibrium states of inhomogeneous equations are stable or asymptotically stable if and only if the same holds for the equilibrium states of the corresponding homogeneous equations.

Example 12.3

Consider again the continuous system

$$\dot{x} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

the stability of which was analyzed earlier in Example 12.1 by using the Lyapunov function method. The characteristic polynomial of the coefficient matrix is

$$\varphi(s) = \det\begin{pmatrix} -s & \omega \\ -\omega & -s \end{pmatrix} = s^2 + \omega^2$$

therefore, the eigenvalues are $\lambda_1 = j\omega$ and $\lambda_2 = -j\omega$. Both eigenvalues have single multiplicities, and Re λ_1 = Re $\lambda_2 = 0$. Hence, the conditions of Part 1 are satisfied, and therefore the equilibrium state is stable. The conditions of Part 2 do not hold. Consequently, the system is not asymptotically stable.

If a time-invariant system is nonlinear, then the Lyapunov method is the most popular choice for stability analysis. If the system is linear, then the direct application of Theorem 12.5 is more attractive, since the eigenvalues of the coefficient matrix $\boldsymbol{A}$ can be obtained by standard methods. In addition, several conditions are known from the literature that guarantee the asymptotic stability of time-invariant discrete and continuous systems even without computing the eigenvalues. For examining asymptotic stability, linearization is an alternative approach to the Lyapunov method as is shown here. Consider the time-invariant continuous and discrete systems

$$\dot{x}(t) = f(x(t))$$

and

$$x(t+1) = f(x(t))$$

Let $J(x)$ denote the Jacobian of $f(x)$, and let $\bar{x}$ be an equilibrium state of the system. It is known that the method of linearization around the equilibrium state results in the time-invariant linear systems

$$\dot{x}_\delta(t) = \mathbf{J}(\bar{x})x_\delta(t)$$

and

$$x_\delta(t+1) = \mathbf{J}(\bar{x})x_\delta(t)$$

where $x_\delta(t) = x(t) - \bar{x}$. It is also known from the theory of difference and ordinary differential equations that the asymptotic stability of the zero vector in the linearized system implies the asymptotic stability of the equilibrium state $\bar{x}$ in the original nonlinear system. The asymptotic stability of the linearized system can be examined by the methodology being discussed above.

For continuous systems, the following results have special importance.

Theorem 12.6

The equilibrium state of a continuous system (Equation 12.4) is asymptotically stable if and only if equation

$$\mathbf{A}^{\mathrm{T}}\mathbf{Q} + \mathbf{Q}\mathbf{A} = -\mathbf{M} \tag{12.6}$$

has positive definite solution $\mathbf{Q}$ with some positive definite matrix $\mathbf{M}$. We note that in practical applications the identity matrix is usually selected for $\mathbf{M}$.

Theorem 12.7

Let $\varphi(\lambda) = \lambda_n + p_{n-1}\lambda^{n-1} + \cdot \cdot \cdot + p_1\lambda + p_0$ be the characteristic polynomial of matrix **A**. Assume that all eigenvalues of matrix **A** have negative real parts. Then $p_i > 0 (i = 0, 1, \ldots, n-1)$.

Corollary. If any of the coefficients p_i is negative or zero, the equilibrium state of the system with coefficient matrix A cannot be asymptotically stable. This result can be used as an initial stability test. However, the conditions of the theorem do not imply that the eigenvalues of A have negative real parts, the corresponding sufficient and necessary conclusions are known as the Routh–Hurwitz stability criteria (see for example, Szidarovszky and Bahill, 1998).

Example 12.4

For matrix

$$A = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$$

the characteristic polynomial is $\varphi(s) = s^2 + \omega^2$. Since the coefficient of s^1 is zero, this system is not asymptotically stable.

The Transfer Function Approach

The transfer function of the continuous system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{y} &= \mathbf{Cx} \end{aligned} \tag{12.7}$$

and that of the discrete system

$$\begin{aligned} x(t+1) &= \mathbf{Ax}(t) + \mathbf{Bu}(t) \\ y(t) &= \mathbf{Cx}(t) \end{aligned} \tag{12.8}$$

have the common form

$$\mathbf{TF}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

If both the input and output are single, then

$$TF(s) = \frac{Y(s)}{U(s)}$$

or in the familiar electrical engineering notation

$$TF(s) = \frac{KG(s)}{1 + KG(s)H(s)} \tag{12.9}$$

where K is the gain term in the forward loop, $G(s)$ represents the dynamics of the forward loop, or the plant, and $H(s)$ models the dynamics in the feedback loop. We note that in the case of continuous systems, s is the variable of the transfer function, and for discrete systems, the variable is denoted by z.

After World War II, systems and control theory flourished. The transfer function representation was the most popular representation for systems. To determine stability of a system we merely had to factor the denominator of the transfer function (Equation 12.9) and see if the poles were in the left half of the complex frequency plane. However, with manual techniques, factoring polynomials of large order was difficult. So, engineers, being naturally lazy people, developed several ways to determine the stability of a system without factoring the polynomials (Dorf, 1992). First, we have methods of Routh and Hurwitz, developed a century ago, that looked at the coefficients of the characteristic polynomial. These methods showed whether the system was stable or not, but they did not show how close the system was to being stable.

What we want to know is, for what value of gain, K, and at what frequency, ω, will the denominator of the transfer function (Equation 12.9) become zero. Or, when $KG(s)H(s) = -1$, meaning when the magnitude of KGH equals 1 with a phase angle of -180 degrees. These parameters can be determined easily with a Bode diagram. Construct a Bode diagram for KHG of the system, look at the frequency where the phase angle equals $-180°$, and look up at the magnitude plot. If it is smaller then 1.0, then the system is stable. If it is larger than

[*AQ1*] 1.0, then the system is unstable. Bode diagram techniques are discussed in Chapter 100.3 [Frequency response methods: Bode diagram approach by Andy Sage].

The quantity $KG(s)H(s)$ is called the open-loop transfer function of the system, because it is the effect that would be encountered by a signal making one loop around the system if the feedback loop were artificially open (Bahill, 1981).

To gain some intuition, think of a closed-loop negative feedback system. Apply a small sinusoid at frequency ω to the input. Assume that the gain around the loop, KGH, is 1 or more, and that the phase angle is $-180°$. The summing junction will flip over the feedback signal and add it to the original signal. The result is a signal that is bigger than what came in. This signal will circulate around this loop, getting bigger and bigger on every loop until the real system no longer matches the model. This is what we call instability.

The question of stability can also be answered with Nyquist diagrams. They are related to Bode diagrams, but they give more information. A simple way to construct a Nyquist diagram is to make a polar plot on the complex frequency plane of the Bode diagram. Simply stated, if this contour encircles the –1 point in the complex frequency plane, then the system is unstable (see Figure 12.2).

The two advantages of the Nyquist technique are (1) in addition to the information of Bode diagrams, there are about a dozen rules that can be used to help construct Nyquist diagrams, and (2) Nyquist diagrams handle bizarre systems better, as is shown in the following rigorous statement of the Nyquist stability criterion. The number of clockwise encirclements minus the number of counter clockwise encirclements of the point $s = -1+j0$ by the Nyquist plot of $KG(s)H(s)$ is equal to the number of poles of $Y(s)/U(s)$ minus the number of poles of $KG(s)H(s)$ in the right half of the s-plane.

The root-locus technique was another popular technique for assessing stability. It furthermore allowed the engineer to see the effects of small changes in the gain, K, on the stability of the system. The root-locus diagram shows the location in the s-plane of the poles of the closed-loop transfer function, $Y(s)/U(s)$. All branches of the root-locus diagram start on poles of open-loop transfer function, KGH, and end either on

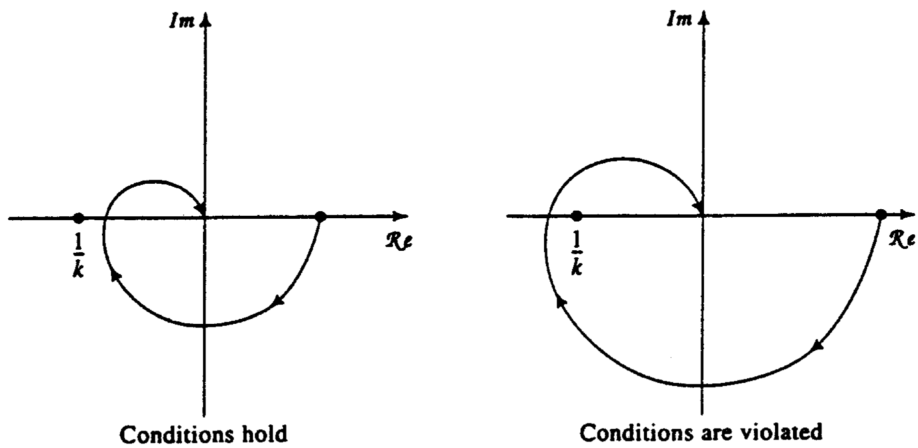

FIGURE 12.2 Illustration of Nyquist stability criteria. (*Source:* F. Szidarovszky and A.T. Bahill, *Linear Systems Theory*, Boca Raton, Fla.: CRC Press, 1998, p. 219. With permission.)

zeros of the open-loop transfer function, *KGH*, or at infinity. There are about a dozen rules to help draw these [AQ1] trajectories. The root-locus technique is discussed in Chapter 100.4 [Root Locus by Ben Kuo].

We consider all these techniques to be old fashioned. They were developed to help answer the question of stability without factoring the characteristic polynomial. However, many computer programs are currently available that factor polynomials. We recommend that engineers merely buy one of these computer packages and find the roots of the closed-loop transfer function to assess the stability of a system.

The poles of a system are defined as all values of s such that sI-A is singular. The poles of the closed-loop transfer function are the same as the eigenvalues of the system: engineers prefer the term *poles* and the symbol s, and mathematicians prefer the term *eigenvalues* and the symbol λ for specific values of s. We will use s for complex frequency and λ for specific values of s.

Sometimes, some poles could be cancelled in the rational function form of $TF(s)$ so that they would not be explicitly shown. However, even if some poles could be cancelled by zeros, we would still have to consider all the poles in the following criteria. The equilibrium state of a continuous System (Equation 12.7) with constant input is stable if all poles of $TF(s)$ have nonpositive real parts and all poles with zero real parts are single. The equilibrium state is asymptotically stable if and only if all poles of $TF(s)$ have negative real parts; that is, all poles are in the left half of the s-plane. Similarly, the equilibrium state of a discrete System (Equation 12.8) with constant input is stable if all poles of $TF(z)$ have absolute values less than or equal to one and all poles with unit absolute values are single. The equilibrium state is asymptotically stable if and only if all poles of $TF(z)$ have absolute values less than one; that is, the poles are all inside the unit circle of the z-plane.

Example 12.5

Consider again the system

$$\dot{x} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which was discussed earlier. Assume that the output equation has the form

$$y = (1, 1)x$$

then

$$TF(s) = \frac{s + \omega}{s^2 + \omega^2}$$

The poles are $+j\omega$ and $-j\omega$, which have zero real parts; that is, they are on the imaginary axis of the s-plane. Consequently, the equilibrium state is stable but not asymptotically stable. A system such as this would produce constant amplitude sinusoids at frequency ω. So it seems natural to assume that such systems would be used to build sinusoidal signal generators and to model oscillating systems. However, this is not the case, because (1) zero resistance circuits are hard to make; therefore, most function generators use other techniques to produce sinusoids; and (2) most real-world oscillating systems (i.e., biological systems) have energy dissipation elements in them.

More generally, real-world function generators are seldom made from closed-loop feedback control systems with -180° of phase shift, because (1) it would be difficult to get a broad range of frequencies and several waveforms from such systems, (2) precise frequency selection would require expensive high-precision components, and (3) it would be difficult to maintain constant frequency in such circuits in the face of changing temperatures and power supply variations. Likewise, closed-loop feedback control systems with -180° of phase shift are not good models for oscillating biological systems, because most biological systems oscillate because of nonlinear network properties.

A special stability criterion for single-input, single-output time-invariant, continuous systems will be introduced next. Consider the system

$$\dot{x} = \mathbf{A}x + \mathbf{b}u \text{ and } y = \mathbf{c}^{\mathrm{T}}x \tag{12.10}$$

where $\mathbf{A}$ is an $n \times n$ constant matrix, and $\mathbf{b}$ and $\mathbf{c}$ are constant n-dimensional vectors. The transfer function of this system is

$$\mathbf{TF}_1(s) = c^{\mathrm{T}}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$$

which is obviously a rational function of s. Now let us add negative feedback around this system so that $u = ky$ where k is a constant. The resulting system can be described by the differential equation

$$\dot{x} = \mathbf{A}x + k\mathbf{b}\mathbf{c}^{\mathrm{T}}x = (\mathbf{A} + k\mathbf{b}\mathbf{c}^{T})x \tag{12.11}$$

The transfer function of this feedback system is

$$TF(s) = \frac{TF_1(s)}{1 - kTF_1(s)} \tag{12.12}$$

To help show the connection between the asymptotic stability of System (Equation 12.10) and System (Equation 12.11), we introduce the following definition.

Definition 12.3

Let $r(s)$ be a rational function of s. Then the locus of points

$$L(r) = \{a + jb | a = \mathrm{Re}(r(jv)),\ b = \mathrm{Im}(r(jv)),\ v \in R\}$$

is called the *response diagram of r.* Note that $L(r)$ is the image of the imaginary line $\mathrm{Re}(s) = 0$ under the mapping r. We shall assume that $L(r)$ is bounded, which is the case if and only if the degree of the denominator is not less than that of the numerator and r has no poles on the line $\mathrm{Re}(s) = 0$.

Theorem 12.8

The Nyquist stability criterion. Assume that TF_1 has a bounded response diagram $L(TF_1)$. If TF_1 has v poles in the right half of the s-plane, where $\mathrm{Re}(s) > 0$, then TF has $\rho + v$ poles in the right half of the s-plane if the point $1/k + j \cdot 0$ is not on $\mathrm{L(TF_1)}$, and $\mathrm{L(TF_1)}$ encircles $1/k + j \cdot 0$ ρ times in the clockwise sense.

Corollary. Assume that System (Equation 12.10) is asymptotically stable with constant input and that $L(TF_1)$ is bounded and traversed in the direction of increasing v and has the point $1/k + j \cdot 0$ on its left. Then the feedback System (Equation 12.11) is also asymptotically stable.

This result has many applications, since feedback systems have a crucial role in constructing stabilizers, observers, and filters for given systems. Figure 12.2 illustrates the conditions of the corollary. The application of this result is especially convenient, if System (Equation 12.10) is given and only appropriate values k of the feedback are to be determined. In such cases, the locus $L(TF_1)$ has to be computed first, and then the region of the appropriate k values can be determined easily from the graph of $L(TF_1)$.

This analysis has dealt with the closed-loop transfer function, whereas the techniques of Bode, root-locus, etc. use the open-loop transfer function. This should cause little confusion as long as the distinction is kept in mind.

12.5 BIBO Stability

In the previous sections, internal stability of time-invariant systems was examined, i.e., the stability of the state was investigated. In this section, the **external stability** of systems is discussed; this is usually called the **BIBO** (*b*ounded-*i*nput, *b*ounded-*o*utput) **stability**. Here we drop the simplifying assumption of the previous section that the system is time-invariant; we will include time variant systems in the following analysis.

Definition 12.4

A system is called BIBO stable if for zero initial conditions, a bounded input always evokes a bounded output.

For continuous systems, a necessary and sufficient condition for BIBO stability can be formulated as follows.

Theorem 12.9

Let $T(t,\tau) = (t_{ij}(t, \tau))$ be the weighing pattern, $\mathbf{C}(t)\boldsymbol{\phi}(t,\tau)\mathbf{B}(\tau)$,of the system. Then the continuous time-variant linear system is BIBO stable if and only if the integral

$$\int_{t_0}^{t} \left| t_{ij}(t,\tau) \right| \mathrm{d}\tau \tag{12.13}$$

is bounded for all $t \geqslant t_0$, i and j.

Corollary. Integrals (Equation 12.13) are all bounded if and only if

$$I(t) = \int_{t_0}^{t} \sum_i \sum_j \left| t_{ij}(t,\tau) \right| \mathrm{d}\tau \tag{12.14}$$

is bounded for $t \geqslant t_0$. Therefore, it is sufficient to show the boundedness of only one integral in order to establish BIBO stability.

The discrete counterpart of this theorem can be given in the following way.

Theorem 12.10

Let $\boldsymbol{T}(t, \tau) = (t_{ij}(t, \tau))$ be the weighing pattern of the discrete linear system. Then it is BIBO stable if and only if the sum

$$I(t) = \sum_{\tau=t_0}^{t-1} \left| t_{ij}(t,\tau) \right| \tag{12.15}$$

is bounded for all $t \geqslant t_0$, i and j.

Corollary. The sums (Equation 12.15) are all bounded if and only if

$$\sum_{\tau=t_0}^{t-1} \sum_i \sum_j \left| t_{ij}(t,\tau) \right| \tag{12.16}$$

is bounded. Therefore, it is sufficient to verify the boundedness of only one sum in order to establish BIBO stability.

Consider next the time-invariant case, when $\mathbf{A}(t) \equiv \mathbf{A}, \mathbf{B}(t) \equiv \mathbf{B}, \mathbf{C}(t) \equiv \mathbf{C}$ From the foregoing theorems and the definition $T(t,\tau)$ we have immediately the following sufficient condition.

Theorem 12.11

Assume that for all eigenvalues λ_i of $\boldsymbol{A}$, $\operatorname{Re}\lambda_i < 0\big(\text{or } |\lambda_i| < 1\big)$. Then the time-invariant linear continuous (or discrete) system is BIBO stable.

BIBO stability is different from stability in the sense of Definition 12.1. For example, a system with a zero eigenvalue might not be BIBO stable, however if the eigenvalue with zero real part is single, then the system still might be stable in the sense of Definition 12.1.

Finally, we note that BIBO stability is not implied by an observation that a certain bounded input generates bounded output. All bounded inputs must generate bounded outputs in order to guarantee BIBO stability.

Adaptive-control systems are time-varying systems. Therefore, it is usually difficult to prove that they are stable. Szidarovszky et al. (1990), however, show a technique for doing this. This result gives a necessary and sufficient condition for the existence of an asymptotically stable model-following adaptive-control system, and in the case of the existence of such systems, they present an algorithm for finding the appropriate feedback parameters.

12.6 Bifurcations

The asymptotic properties of the equilibrium state of any dynamic system depend on the particular values of the model parameters. For certain values, the system might be asymptotically stable and with a sudden change of one or more parameter values this stability disappears. Such sudden change in asymptotical behavior is called bifurcation. As a simple illustration, consider the following extension of Example 12.1.

Example 12.6

Consider the system

$$\dot{x} = \begin{pmatrix} \delta & \omega \\ -\omega & \delta \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

in which $\omega > 0$ and δ is a real parameter. The characteristic polynomial of the system can be written as

$$\lambda^2 - 2\lambda\delta + (\delta^2 + \omega^2) = 0$$

so the eigenvalues are

$$\lambda_1 = \delta + \mathrm{j}\omega \text{ and } \lambda_2 = \delta - \mathrm{j}\omega$$

If $\delta < 0$, then both eigenvalues have negative real parts implying asymptotical stability. If $\delta > 0$, then the real part of the eigenvalues becomes positive, so the equilibrium becomes unstable. So at $\delta = 0$ bifurcation occurs with an eigenvalues with zero-real part.

Parameter δ is called the **bifurcation parameter**, since we examine the change of stability behavior as a function of the change of its value. The eigenvalues also depend on the value of the bifurcation parameter. If the real part of the derivative of the pure complex eigenvalue with respect to the bifurcation parameter is nonzero, then *Hopf-bifurcation* occurs, which guarantees the birth of limit cycles around the equilibrium. Other bifurcation types and their conditions with applications are discussed for example, in Guckenheimer and Holmes (1983).

12.7 Physical Examples

In this section, we show some examples of stability analysis of physical systems.

1. Consider a simple *harmonic oscillator* constructed of a mass and an ideal spring. Its dynamic response is summarized with

$$\dot{x} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

   In Example 12.3, we showed that this system is stable but not asymptotically stable. This means that if we leave it alone in its equilibrium state, it will remain stationary, but if we jerk on the mass it will oscillate forever. There is no damping term to remove the energy, so the energy will be transferred back and forth between potential energy in the spring and kinetic energy in the moving mass. A good approximation of such a harmonic oscillator is the pendulum clock. The more expensive it is (i.e., the smaller the damping), the less often we have to wind it (i.e., add energy).
2. A *linear second-order electrical system* composed of a series connection of an input voltage source, an inductor, a resistor, and a capacitor, with the output defined as the voltage across the capacitor, can be characterized by the second-order equation

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{1}{LCs^2 + RCs + 1}$$

   For convenience, let us define

$$\omega_n = \sqrt{\frac{1}{LC}} \text{ and } \zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$$

   With these parameters the transfer function becomes

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

   Is this system stable? The roots of the characteristic equation are

$$\lambda_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$$

   If $\zeta > 0$, the poles are in the left half of the s-plane, and therefore the systems is asymptotically stable. If $\zeta = 0$, as in the previous example, the poles are on the imaginary axis; therefore, the system is stable but not asymptotically stable. If $\zeta < 0$ the poles are in the right half of the s-plane and the system is unstable.
3. An *electrical system* is shown in Figure 12.3. Simple calculation shows that by introducing the variables

$$x_1 = i_L,\ x_2 = v_c, \quad \text{and} \quad u = v_s$$

   the system can be described by the differential equations

$$\dot{x}_1 = -\frac{R_1}{L}x_1 - \frac{1}{L}x_2 + \frac{1}{L}u$$

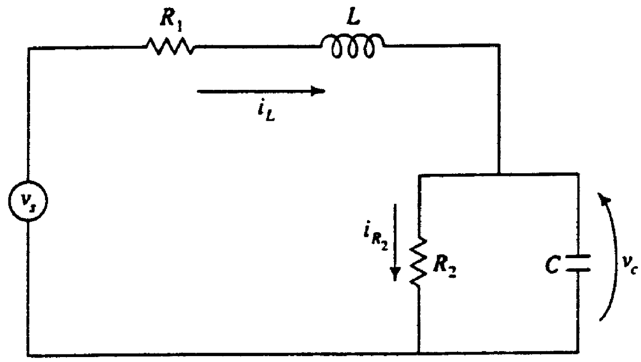

FIGURE 12.3 A simple electrical system. (*Source:* F. Szidarovszky and A.T. Bahill, *Linear Systems Theory*, Boca Raton, Fla.: CRC Press, 1998, p. 158. With permission.)

$$\dot{x}_2 = \frac{1}{C}x_1 - \frac{1}{CR_2}x_2$$

The characteristic equation has the form

$$\left(-s - \frac{R_1}{L}\right)\left(-s - \frac{1}{CR_2}\right) + \frac{1}{LC} = 0$$

which simplifies as

$$s^2 + s\left(\frac{R_1}{L} + \frac{1}{CR_2}\right) + \left(\frac{R_1}{LCR_2} + \frac{1}{LC}\right) = 0$$

Since R_1, R_2, L, and C are positive numbers, the coefficients of this equation are all positive. The constant term equals $\lambda_1\lambda_2$, and the coefficient of s^1 is $-(\lambda_1 + \lambda_2)$. Therefore

$$\lambda_1 + \lambda_2 < 0 \quad \text{and} \quad \lambda_1\lambda_2 > 0$$

If the eigenvalues are real, then these relations hold if and only if both eigenvalues are negative. If they were positive, then $\lambda_1 + \lambda_2 > 0$. If they had different signs, then $\lambda_1\lambda_2 < 0$. Furthermore, if at least one eigenvalue is zero, then $\lambda_1\lambda_2 = 0$. Assume next that the eigenvalues are complex:

$$\lambda_{1,2} = \mathrm{Re}\, s \pm \mathrm{j}\, \mathrm{Im}\, s$$

Then

$$\lambda_1 + \lambda_2 = 2\, \mathrm{Re}\, s$$

and

$$\lambda_1\lambda_2 = (\mathrm{Re}\, s)^2 \pm (\mathrm{Im}\, s)^2$$

Hence $\lambda_1 + \lambda_2 < 0$ implies that $\mathrm{Re}\ s < 0$.

In summary, the system is asymptotically stable, since in both the real and complex cases the eigenvalues have negative values and negative real parts, respectively.

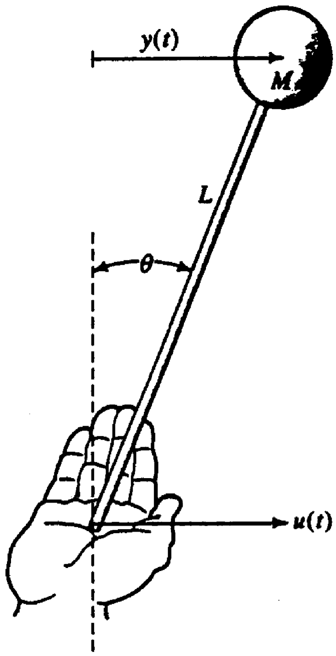

FIGURE 12.4 Stick balancing. (*Source:* F. Szidarovszky and A.T. Bahill, *Linear Systems Theory*, Boca Raton, Fla.: CRC Press, 1998, p. 165. With Permission.)

4. The classical *stick-balancing* problem shown in Figure 12.4. Simple analysis shows that $y(t)$ satisfies the second-order equation

$$\ddot{y} = \frac{g}{L}(y - u)$$

If one selects $L = 1$, then the characteristic equation has the form

$$s^2 - g = 0$$

So the eigenvalues are

$$\lambda_{1,2} = \pm\sqrt{g}$$

One is in the right half of the s-plane and the other is in the left half of the s-plane, so the system is unstable. This instability is understandable, since without an intelligent input to control the system, if the stick is not upright with zero velocity, it will fall over.

5. A simple transistor circuit can be modeled as shown in Figure 12.5. The state variables are related to the input and output of the circuit: The base current, i_b, is x_1 and the output voltage, v_{out}, is x_2. Therefore,

$$\dot{x} = \begin{pmatrix} -\dfrac{h_{ie}}{L} & 0 \\ \dfrac{h_{fe}}{C} & 0 \end{pmatrix} x + \begin{pmatrix} \dfrac{1}{L} \\ 0 \end{pmatrix} e_s \quad \text{and} \quad c^{\mathrm{T}} = (0, 1)$$

The **A** matrix looks strange with a column of all zeros, and indeed the circuit does exhibit odd behavior. For example, as we will show, there is no equilibrium state for a unit step input of e_s. This is reasonable, however, because the model is for mid-frequencies, and a unit step does not qualify. In response to a unit

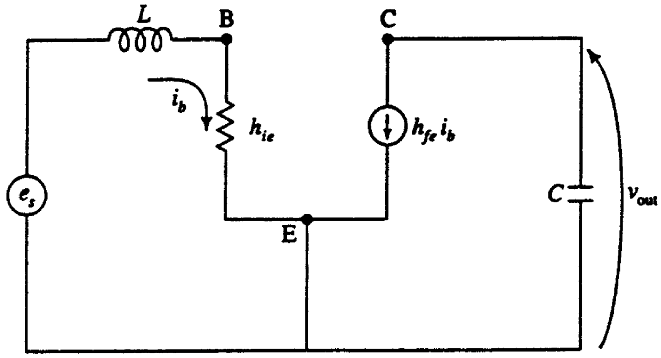

FIGURE 12.5 A model for a simple transistor circuit. (*Source:* F. Szidarovszky and A.T. Bahill, *Linear Systems Theory*, Boca Raton, Fla.: CRC Press, 1998, p. 160. With Permission.)

step the output voltage will increase linearly until the model is no longer valid. If e_s is considered the input, the system is

$$\dot{x} = \begin{pmatrix} -\frac{h_{ie}}{L} & 0 \\ \frac{h_{fe}}{C} & 0 \end{pmatrix} x + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} u$$

If $u(t) \equiv 1$, then at the equilibrium state

$$\begin{pmatrix} -\frac{h_{ie}}{L} & 0 \\ \frac{h_{fe}}{C} & 0 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

That is,

$$-\frac{h_{ie}}{L}\bar{x}_1 + \frac{1}{L} = 0$$

$$\frac{h_{fe}}{C}\bar{x}_1 = 0$$

Since $h_{fe}/C \neq 0$, the second equation implies that $\bar{x}_1 = 0$, and by substituting this value into the first equation we get the obvious contradiction $1/L = 0$. Hence, with nonzero constant input *no* equilibrium state exists.

Let us now investigate the stability of this system. First let $\tilde{x}(t)$ denote a fixed trajectory of this system, and let $\boldsymbol{x}(t)$ be an arbitrary solution. Then the difference $\delta x(t) = x(t) - \tilde{x}(t)$ satisfies the homogenous equation

$$\delta\dot{x} = \begin{pmatrix} -\frac{h_{ie}}{L} & 0 \\ \frac{h_{fe}}{C} & 0 \end{pmatrix} \delta x$$

This system has an equilibrium $\boldsymbol{\delta x}(t) = 0$. Next, the stability of this equilibrium is examined by solving for the poles of the transfer function. The characteristic equation is

$$\det\begin{pmatrix} -\dfrac{h_{ie}}{L} - s & 0 \\ \dfrac{h_{fe}}{C} & -s \end{pmatrix} = 0$$

which can be simplified as

$$s^2 + s\frac{h_{ie}}{L} + 0 = 0$$

The roots are

$$\lambda_1 = 0 \text{ and } \lambda_2 = -\frac{h_{ie}}{L}$$

Therefore, the system is stable but not asymptotically stable. This stability means that for small changes in the initial state the entire trajectory $x(t)$ remains close to $\tilde{x}(t)$.

Defining Terms

Asymptotic stability: An equilibrium state $\bar{x}$ of a system is asymptotically stable if, in addition to being stable, there is an $\varepsilon > 0$ such whenever $\|\bar{x} - x_0\| < \varepsilon$, then $x(t) \rightarrow \bar{x}$ as $t \rightarrow \infty$. A system is asymptotically stable if all the poles in a closed-loop transfer function are in the left half of the s-plane (inside the unit circle of the z-plane for discrete systems).

BIBO stability: A system is BIBO stable if for zero initial conditions any bounded input always evokes a bounded output.

Bifurcation: If a sudden change of a model parameter value results in a change of the asymptotic behavior of the system. This model parameter is called the *bifurcation parameter*, and its specific value where the change occurs, is called the *critical value.*

External stability: Stability concepts related to the input-ouput behavior of the system.

Global asymptotic stability: An equilibrium state $\bar{x}$ of a system is globally asymptotically stable if it is stable and with arbitrary initial state $x_0 \in X, x(t) \rightarrow \bar{x}$ as $t \rightarrow \infty$.

Instability: An equilibrium state of a system is unstable if it is not stable. A system is unstable if at least one pole of the closed-loop transfer is in the right half of the s-plane (outside the unit circle of the z-plane for discrete systems). A system might be unstable if poles with zero-real parts (with unit absolute values) are multiple.

Internal stability: Stability concepts related to the state of the system.

Stability: An equilibrium state $\bar{x}$ of a system is stable if there is an $\varepsilon_0 > 0$ with the following property: for all ε_1, $0 < \varepsilon_1 < \varepsilon_0$, there is an $\varepsilon > 0$ such that if $\|\bar{x} - x_0\| < \varepsilon$, then $|1\bar{x} - 1x(t)| < \varepsilon_1$.for all $t > t_0$ A system is stable if the poles of its closed-loop transfer function are (1) in the left half of the complex frequency plane, called the s-plane (inside the unit circle of the z-plane for discrete systems), or (2) on the imaginary axis, and all of the poles on the imaginary axis are single (on the unit circle and such poles are single for discrete systems). Stability for a system with repeated poles on the $j\omega$ axis (the unit circle) is complicated and is examined in the discussion after Theorem 12.5. In the electrical engineering literature, this definition of stability is sometimes called *marginal stability* and sometimes *stability in the sense of Lyapunov.*

References

A.T. Bahill, *Bioengineering: Biomedical, Medical and Clinical Engineering,* Englewood Cliffs, N.J.: Prentice-Hall, 1981, pp. 214–215, 250–252.

R.C. Dorf, *Modern Control Systems*, 6th Edition, Reading, Mass.: Addison-Wesley, 1992.

J. Guckenheimer and P. Holmes, *Nonlinear Oscillations*, *Dynamic Systems and Bifurcation of Vector Fields,* Berlin/Heidelberg/New York: Springer-Verlag, 1983.

M. Jamshidi, M. Tarokh, and B. Shafai, *Computer-Aided Analysis and Design of Linear Control Systems,* Englewood Cliffs, N.J.: Prentice-Hall, 1992.

A.M. Lyapunov, "Problème général de la stabilité du mouvement" (in French), *Ann. Fac. Sci. Toulouse*, vol. 9, 1907, pp. 203–474. Reprinted in *Ann. Math. Study,* no. 17, 1949, Princeton University Press.

F. Szidarovszky and A.T. Bahill, *Linear Systems Theory,* 2^{nd} Edition, Boca Raton, Fla.: CRC Press, 1998.

F. Szidarovszky, A.T. Bahill and S. Molnar, "On stable adaptive control systems," *Pure Math. and Appl.,* vol. 1, ser. B, no. 2–3, pp. 115–121, 1990.

Further Information

For further information, consult the textbooks *Modern Control Systems* by Dorf (1992) or *Linear Systems Theory* by Szidarovszky and Bahill (1998).

Author Queries

BOOK NUMBER: BK10513/10526

CHAPTER: Stability Analysis

Q1 Please check chapter number and update accordingly.