# Optimizing Baseball and Softball Bats

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**Abstract** Collisions between baseballs, softballs and bats are complex and therefore their models are complex. One purpose of this paper is to show how complex these collisions can be, while still being modeled using only Newton's principles and the conservation laws of physics. This paper presents models for the speed and spin of balls and bats. These models and equations for bat-ball collisions are intended for use by high school and college physics students, engineering students and most importantly students of the science of baseball. Unlike in previous papers, these models use only simple Newtonian principles to explain simple collision configurations.

## 1 Précis of This Endeavor

This paper has two primary purposes: first, to help a batter select or create an optimal baseball or softball bat and second, to create models for bat-ball collisions using only fundamental principles of Newtonian mechanics (Table 1). We note that force, velocity, acceleration, impulse and momentum are all vector quantities, although we do not specifically mark them as such.

Newton's principles of motion are idealized as

I. *Inertia.* Every object either remains at rest or continues to move at a constant velocity, unless acted upon by an external force.

$$\sum F = 0 \quad \Leftrightarrow \quad dv/dt = 0$$

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**Table 1** List of variables, inputs, parameters, constants and their abbreviations

| Symbol: This table is arranged alphabetically by the symbol | Abbreviation ball = 1 bat = 2 before = b after = a | Description | Typical values for a C243 pro stock wooden bat and a professional major-league baseball player | |
|---|---|---|---|---|
| | | | SI units | Baseball units |
| $\beta_{\text{bat-knob}}$ | $\beta$ | Angular velocity of the bat about the knob | rad/s | rpm |
| *CoE* | | Conservation of energy | Joules | |
| *CoM* | | Conservation of momentum | kg m/s | |
| *CoAM* | | Conservation of angular momentum | $\text{kg m}^2/\text{s}$ | |
| *CoR* | | Coefficient of restitution of a bat-ball collision | 0.55 | 0.55 |
| $d_{\text{bat}}$ | | Length of the bat | 0.861 m | 34 in. |
| $d_{\text{bat-cm-ss}}$ | $d_{\text{cm-ss}}$ $d$ | Distance from the center of mass to the sweet spot, which we define as the Center of Percussion | 0.134 m | 5.3 in. |
| $d_{\text{bat-knob-cm}}$ | $d_{\text{kcm}}$ | Distance from the center of the knob to the center of mass | 0.569 m | 22.4 in. |
| $d_{\text{bat-knob-ss}}$ | $d_{\text{kss}}$ | Distance from the center of the knob to the sweet spot | 0.705 m | 27.8 in. |
| $d_{\text{bat-pivot-cm}}$ | | Distance from the pivot point to the center of mass | 0.416 m | 16.4 in. |
| $d_{\text{spine-cm}}$ | | Distance from the batter's spine to the center of mass of the bat, an experimentally measured value | 1.05 m | 41 in. |
| $d_{\text{bat-ss-end}}$ | | Distance from the sweet spot to the barrel end of the bat | 0.149 m | 5.9 in. |
| g | | Earth's gravitational constant (at the UofA) | 9.718 m/s | |
| $I_{\text{ball}}$ | $I_1$ | Moment of inertia of the ball with respect to its center of mass | 0.000079 kg $\text{m}^2$ | |
| $I_{\text{bat-cm}}$ | $I_2$ | Moment of inertia of the bat with respect to rotations about its center of mass | 0.048 kg $\text{m}^2$ | |
| $I_{\text{bat-knob}}$ | $I_{\text{k}}$ | Moment of inertia of the bat with respect to rotations about the knob | 0.341 kg $\text{m}^2$ | |
| $I_{\text{bat-pivot}}$ | | Moment of inertia of the bat with respect to the pivot point between the hands | 0.208 kg $\text{m}^2$ | |

(continued)

**Table 1** (continued)

| Symbol: This table is arranged alphabetically by the symbol | Abbreviation ball = 1 bat = 2 before = b after = a | Description | Typical values for a C243 pro stock wooden bat and a professional major-league baseball player | |
|---|---|---|---|---|
| | | | SI units | Baseball units |
| $KE_{\text{before}}$ | | Kinetic energy of the bat and the ball before the collision | 375 J | |
| $KE_{\text{after}}$ | | Kinetic energy of the bat and the ball after the collision | 216 J | |
| $KE_{\text{lost}}$ | | Kinetic energy lost or transformed in the collision | 158 J | |
| $m_{\text{ball}}$ | $m_1$ | Mass of the baseball | 0.145 kg | 5.125 oz |
| $m_{\text{bat}}$ | $m_2$ | Mass of the bat | 0.905 kg | 32 oz |
| $\bar{m}$ | | $\bar{m} = \frac{m_{\text{ball}} m_{\text{bat}}}{m_{\text{ball}} + m_{\text{bat}}}$ | 0.125 kg | 4.4 oz |
| $\mu_f$ | | Dynamic coefficient of friction for a ball sliding on a wooden bat | 0.5 | |
| $r_{\text{ball}}$ | $r_1$ | Radius of the baseball | 0.037 m | 1.45 in. |
| $r_{\text{bat}}$ | $r_2$ | Maximum allowed radius of the bat | 0.035 m | 1.37 in. |
| *pitch speed* | | Speed of the ball at the pitcher's release point | −46 | −92[a] mph |
| $v_{\text{ball-before}}$ | $v_{1\text{b}}$ | Velocity of the ball immediately before the collision, 90% of pitch speed | −37 m/s | −83[a] mph |
| $v_{\text{ball-before-Norm}}$ | $v_{1\text{bN}}$ | Normal component of curveball velocity before collision, $v_{\text{ball-before}} \cos 6^\circ$ | −36.8 m/s | −82.3 mph |
| $v_{\text{ball-before-Tan}}$ | $v_{1\text{bT}}$ | Tangential component of curveball velocity before collision, $v_{\text{ball-before}} \sin 6^\circ$ | −3.9 m/s | −8.7 mph |
| $v_{\text{ball-after}}$ | $v_{1\text{a}}$ | Velocity of the ball after the collision, often called the launch speed or the *batted-ball speed*. | 41.6 m/s | 93 mph |
| $v_{\text{bat}}$ | $v_2$ | Velocity of the bat. If a specific place or time is intended then the subscript may contain cm (center of mass), ss (sweet spot), before (b) or after (a). | | |

(continued)

**Table 1** (continued)

| Symbol: This table is arranged alphabetically by the symbol | Abbreviation ball = 1 bat = 2 before = b after = a | Description | Typical values for a C243 pro stock wooden bat and a professional major-league baseball player | |
|---|---|---|---|---|
| | | | SI units | Baseball units |
| $v_{\text{bat-cm-before}}$ | $v_{2\text{cmb}}$ | Velocity of the center of mass of the bat before the bat-ball collision. | 23 m/s | 51 mph |
| $v_{\text{bat-cm-after}}$ | $v_{2\text{cma}}$ | Velocity of the center of mass of the bat after the collision. | 10.4 m/s | 23 mph |
| $v_{\text{bat-ss-before}}$ | $v_{2\text{ssb}}$ | Velocity of the sweet spot of the bat before the collision. | 26 m/s | 58[a] mph |
| $v_{\text{bat-ss-after}}$ | $v_{2\text{ssa}}$ | Velocity of the sweet spot of the bat after the collision. | 12 m/s | 27 mph |
| $\omega_{\text{ball-before}}$ | $\omega_{1\text{b}}$ | Angular velocity of the ball about its center of mass before the collision. This spin rate depends on the particular type of pitch. | ±209 rad/s | ±2000 rpm |
| $\omega_{\text{ball-after}}$ | $\omega_{1\text{a}}$ | Angular velocity of the ball about its center of mass after the collision | ±209 rad/s | ±2000 rpm |
| $\omega_{\text{bat-before}}$ | $\omega_{2\text{b}}$ | Angular velocity of the bat about its center of mass before the collision | Near zero | |
| $\omega_{\text{bat-after}}$ | $\omega_{2\text{a}}$ | Angular velocity of the bat about its center of mass after the collision | −32 rad/s | −303 rpm |
| $\omega_{\text{spine-before}}$ | | Angular velocity of the batter's arms and the bat about the spine | 21 rad/s | 201 rpm |

[a]The equations of this paper concern variables right before and right after the collision, not at other times. For example, a pitcher could release a fastball with a speed of 92 mph, by the time it got to the collision zone it would have slowed down by 10% to 83 mph. Therefore, in our simulations we used 83 mph for $v_{\text{ball-before}}$

II. *Impulse and Momentum*. The rate of change of momentum of a body is directly proportional to the force applied and is in the direction of the applied force.

$$F = \frac{d(mv)}{dt} \Leftrightarrow F = ma$$

Stated differently, the change of momentum of a body is proportional to the impulse applied to the body, and has a direction along the straight line upon which that impulse is applied. An impulse $J$ occurs when a force $F$ acts over an interval of time $\Delta t$, and it is given by $J = \int_{\Delta t} F dt$. Since force is the time derivative of momentum, it follows that $J = \Delta p = m\Delta v$. Applying an impulse changes the momentum.

III. *Action/reaction.* For every action there is an equal and opposite reaction.
IV. *Restitution.* The ratio of the relative speeds after and before the collision is defined as the coefficient of restitution (*CoR*). The relative speed of two objects after a collision is a fixed fraction of the relative speed before the collision, regardless of whether one object or the other is initially at rest or the objects are approaching each other. The *CoR* models the energy lost in a collision.

In this paper, we will use these four principles of Newton. We will also use the overarching conservation laws that state, *energy, linear momentum and angular momentum cannot be created or destroyed.* These laws are more general than the principles and apply in all circumstances.

## 2 Bat-Ball Collisions

In this paper, we are modeling a point in time right before the bat-ball collision and its relationship with another point just after the collision. We are not modeling the behavior (1) during the collision, (2) long before the collision (the pitched ball) or (3) long after the collision (the batted-ball). The flight of the ball has been modeled by Bahill et al. (2009).

My model is for a head-on collision at the sweet spot (ss) of the bat, which I define to be the Center of Percussion (Bahill 2004). Figure 1 is a diagram of such a collision. All figures are drawn for a right-handed batter. This type of analysis was done by Watts and Bahill (1990, 2000). It would produce a "line drive" back to the pitcher.

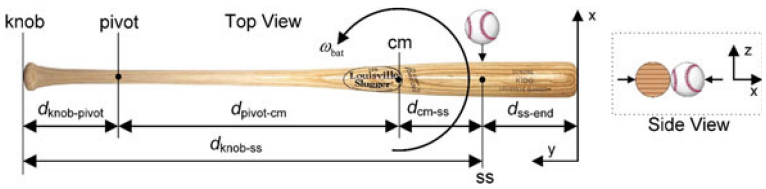


**Fig. 1** Model for a collision at the sweet spot (ss) of the bat

# 3 Equations for Bat-Ball Collisions

## 3.1 Collisions at the Center of Mass

The literature is abound with linear collisions at the center of mass of an object. In these, kinetic energy is transformed into heat in the ball, vibrations in the bat, acoustic energy in the "crack of the bat" and deformations of the bat or ball. The Coefficient of Restitution (*CoR*) models the energy that is transformed in a frictionless head-on collision between two objects. The equation for the kinetic energy lost in a head-on bat-ball collision at the center of mass (Dadouriam 1913, Eq. (XI), p. 248; Ferreira da Silva 2007, Eq. 23; Brach 2007, Eq. 3.7) is

$$\begin{aligned} KE_{lost} &= \frac{\bar{m}}{2}(\text{collision velocity})^2\left(1 - CoR_{1b}^2\right) \text{ where } \bar{m} = \frac{m_{\text{ball}} m_{\text{bat}}}{m_{\text{ball}} + m_{\text{bat}}} \\ KE_{lost} &= \frac{\bar{m}}{2}\left(v_{\text{bat}-\text{cm}-\text{before}} - v_{\text{ball}-\text{before}}\right)^2\left(1 - CoR_{1b}^2\right) \end{aligned} \tag{1}$$

## 3.2 Collisions at the Sweet Spot

### 3.2.1 Coordinate System

We use a right-handed coordinate system with the x-axis pointing from home plate to the pitching rubber, the y-axis points from first base to third base, and the z-axis points straight up. A torque rotating from the x-axis to the y-axis would be positive upward. Over the plate, the ball comes downward at a 10° angle and the bat usually moves upward at about 10°, so later the z-axis will be rotated back 10°.

### 3.2.2 Assumptions

We made the following assumptions:

A1. We assumed a head-on collision at the sweet spot of the bat.
A2. We neglected permanent deformations of the bat and ball.
A3. We assumed that there were no tangential forces during the collision.
A4. In this paper, we did not model the moment of inertia of the batter's arms.
A5. Collisions at the Center of Percussion produce a rotation about the center of mass, but no translation of the bat.
A6. The collision duration is short, for example, one millisecond.
A7. Because the collision duration is short and the swing is level, we ignored the effects of gravity *during* the collision.
A8. The Coefficient of Restitution (*CoR*) for a baseball wooden-bat collision at major-league speeds is about 0.55.

A9. The dynamic coefficient of friction has been measured by Bahill at $\mu_f = 0.5$. This agrees with measurements by Sawicki et al. (2003) and Cross and Nathan (2006).

A10. Air density affects the flight of the batted-ball. And air density is inversely related to altitude, temperature and humidity, and is directly related to barometric pressure. Of these four, altitude is the most important factor (Bahill et al. 2009). We did not consider these four parameters in this paper, because they are for the flight of the ball, not the collision.

### 3.2.3 Experimental Validation Data

The experimental data in Table 1 are based on the following assumptions. The batter is using a Louisville Slugger C243 wooden bat and is hitting a regulation major-league baseball. The ball speed at the plate is −83 mph. The velocity of the sweet spot of the bat is 58 mph: this is the average value of the data collected from 28 San Francisco Giants measured by Bahill and Karnavas (1991). These velocities produce a *CoR* of 0.55 and a batted-ball speed of 97 mph, as will be shown in Table 7. Using an ideal launch angle of 31°, we find a batted-ball spin of −2100 rpm (Baldwin and Bahill 2004). With these values, the ball would travel 350 feet, which could produce a home run in all major-league stadiums.

### 3.2.4 The Model

The model of this paper is for a collision at the sweet spot of the bat with spin on the pitch. The model for the movement of the bat is a translation and a rotation about the center of mass. It has five equations and five unknowns, which are shown in Table 2.

Definition of Variables

To visualize these variables please refer to Fig. 2.

Inputs $v_{\text{ball}-\text{before}}$, $\omega_{\text{ball}-\text{before}}$, $v_{\text{bat}-\text{cm}-\text{before}}$, $\omega_{\text{bat}-\text{before}}$ and *CoR*

$v_{\text{ball}-\text{before}}$ is the linear velocity of the *ball* in the x-direction before the collision.

$\omega_{\text{ball}-\text{before}}$ is the angular velocity of the *ball about its center of mass* before the collision.

$v_{\text{bat}-\text{cm}-\text{before}}$ is the linear velocity of the *center of mass of the bat* in the x-direction before the collision.

$\omega_{\text{bat}-\text{before}}$ is the angular velocity of the *bat about its center of mass* before the collision.

**Table 2** The model has five equations and five unknowns

| | |
|---|---|
| Inputs | $v_{\text{ball-before}}$, $\omega_{\text{ball-before}}$, $v_{\text{bat-cm-before}}$, $\omega_{\text{bat-before}}$ and $CoR$ |
| Outputs (unknowns) | $v_{\text{ball-after}}$, $\omega_{\text{ball-after}}$, $v_{\text{bat-cm-after}}$, $\omega_{\text{bat-after}}$, and $KE_{\text{lost}}$ |
| *Equations* | |
| Conservation of energy, Eq. (2) | $\frac{1}{2}m_{\text{ball}}v_{\text{ball-before}}^2 + \frac{1}{2}I_{\text{ball}}\omega_{\text{ball-before}}^2 + \frac{1}{2}m_{\text{bat}}v_{\text{bat-cm-before}}^2 + \frac{1}{2}I_{\text{bat}}\omega_{\text{bat-before}}^2$ $= \frac{1}{2}m_{\text{ball}}v_{\text{ball-after}}^2 + \frac{1}{2}I_{\text{ball}}\omega_{\text{ball-after}}^2 + \frac{1}{2}m_{\text{bat}}v_{\text{bat-cm-after}}^2 + \frac{1}{2}I_{\text{bat}}\omega_{\text{bat-after}}^2 + KE_{\text{lost}}$ |
| Conservation of linear momentum, Eq. (3) | $m_{\text{ball}}v_{\text{ball-before}} + m_{\text{bat}}v_{\text{bat-cm-before}} = m_{\text{ball}}v_{\text{ball-after}} + m_{\text{bat}}v_{\text{bat-cm-after}}$ |
| Definition of $CoR$, Eq. (4) | $CoR_{2\text{b}} = -\frac{v_{\text{ball-after}} - v_{\text{bat-cm-after}} - d_{\text{cm-ss}}\omega_{\text{bat-after}}}{v_{\text{ball-before}} - v_{\text{bat-cm-before}} - d_{\text{cm-ss}}\omega_{\text{bat-before}}}$ |
| Newton's second law, Eq. (5) | $d_{\text{cm-ss}}m_{\text{ball}}(v_{\text{ball-after}} - v_{\text{ball-before}}) = -I_{\text{bat}}(\omega_{\text{bat-after}} - \omega_{\text{bat-before}})$ |
| Conservation of angular momentum, Eq. (6s) | $L_{\text{ball-before}} + L_{\text{bat-before}} = L_{\text{ball-after}} + L_{\text{bat-after}}$ $m_1 v_{1\text{b}} d + I_1\omega_{1b} + m_1\omega_{1b}d^2 + I_2\omega_{2b} = m_1 v_{1\text{a}} d + I_1\omega_{1a} + m_1\omega_{1a}d^2 + I_2\omega_{2\text{a}}$ |

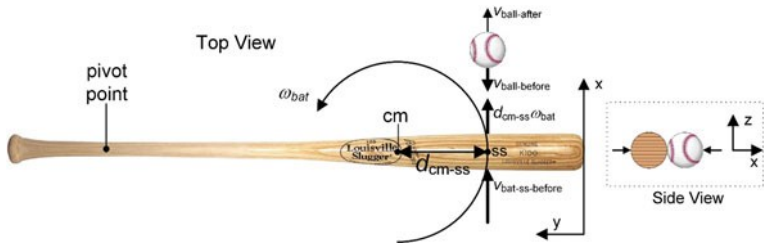


**Fig. 2** This figure shows $v_{\text{ball}-\text{before}}$, $v_{\text{bat}-\text{cm}-\text{before}}$, $v_{\text{ball}-\text{after}}$ and $d_{\text{cm}-\text{ss}}\omega_{\text{bat}}$, which are used to define the coefficient of restitution

$CoR_{2b}$ is the coefficient of restitution.

Outputs $v_{\text{ball}-\text{after}}$, $\omega_{\text{ball}-\text{after}}$, $v_{\text{bat}-\text{ss}-\text{after}}$, $\omega_{\text{bat}-\text{after}}$ and $KE_{\text{lost}}$

$v_{\text{ball}-\text{after}}$ is the linear velocity of the *batted-ball* in the x-direction after the collision.

$\omega_{\text{ball}-\text{after}}$ is the angular velocity of the *ball about its center of mass* after the collision.

$v_{\text{bat}-\text{ss}-\text{after}}$ is the linear velocity of the *sweet spot* of the bat in the x-direction after the collision.

$\omega_{\text{bat}-\text{after}}$ is the angular velocity of the bat *about its center of mass* after the collision.

$KE_{\text{lost}}$ is the kinetic energy lost or transformed in the collision.

We want to solve for $v_{\text{ball}-\text{after}}$, $\omega_{\text{ball}-\text{after}}$, $v_{\text{bat}-\text{cm}-\text{after}}$, $\omega_{\text{bat}-\text{after}}$ and $KE_{\text{lost}}$.

We will use the following fundamental equations of physics: Conservation of Energy, Conservation of Linear Momentum, the Definition of Kinematic *CoR*, Newton's Second Principle and the Conservation of Angular Momentum.

*Condensing the Notation for the Equations*

First, we want to simplify our notation. We will now make the following substitutions. These abbreviations are contained in Table 1, but for convenience, we repeat them here.

$$d_{\text{cm}-\text{ss}} = d$$
$$I_{\text{bat}} = I_2$$
$$m_{\text{ball}} = m_1$$
$$m_{\text{bat}} = m_2$$
$$v_{\text{ball}-\text{before}} = v_{1\text{b}}$$
$$v_{\text{ball}-\text{after}} = v_{1\text{a}}$$
$$v_{\text{bat}-\text{cm}-\text{before}} = v_{2\text{b}}$$
$$v_{\text{bat}-\text{cm}-\text{after}} = v_{2\text{a}}$$
$$\omega_{\text{bat}-\text{before}} = \omega_{2\text{b}}$$
$$\omega_{\text{bat}-\text{after}} = \omega_{2\text{a}}$$

These substitutions produce the following equations.

*Conservation of Energy*

The law of conservation of energy states that energy will not be create or destroyed.

$$\frac{1}{2}m_{\text{ball}}v_{\text{ball}-\text{before}}^2+\frac{1}{2}I_{\text{ball}}\omega_{\text{ball}-\text{before}}^2+\frac{1}{2}m_{\text{bat}}v_{\text{bat}-\text{cm}-\text{before}}^2+\frac{1}{2}I_{\text{bat}}\omega_{\text{bat}-\text{before}}^2$$
$$=\frac{1}{2}m_{\text{ball}}v_{\text{ball}-\text{after}}^2+\frac{1}{2}I_{\text{ball}}\omega_{\text{ball}-\text{after}}^2+\frac{1}{2}m_{\text{bat}}v_{\text{bat}-\text{cm}-\text{after}}^2+\frac{1}{2}I_{\text{bat}}\omega_{\text{bat}-\text{after}}^2+KE_{\text{lost}} \tag{2}$$

$$m_1v_{1\text{b}}^2+m_2v_{2\text{b}}^2+I_2\omega_{2\text{b}}^2=+m_1v_{1\text{a}}^2+m_2v_{2\text{a}}^2+I_2\omega_{2\text{a}}^2+2KE_{\text{lost}} \tag{2s}$$

In the label (3s), "s" stands for short.

*Conservation of Linear Momentum*

The law of conservation of linear momentum states that linear momentum will be conserved in a collision if there are no external forces. We will approximate the bat's motion before the collision with the tangent to the curve of its arc as shown in Fig. 2. For a collision anywhere on the bat, every point on the bat has the same angular velocity, but the linear velocities will be different, which means that $v_{\text{bat}-\text{before}}$ is a combination of translations and rotations unique for each point on the bat. Conservation of momentum in the direction of the x-axis states that the momentum before plus the external impulse will equal the momentum after the collision. There are no external impulses during the bat-ball collision: therefore, this is the equation for Conservation of Linear Momentum

$$m_{\text{ball}}v_{\text{ball}-\text{before}}+m_{\text{bat}}v_{\text{bat}-\text{cm}-\text{before}}=m_{\text{ball}}v_{\text{ball}-\text{after}}+m_{\text{bat}}v_{\text{bat}-\text{cm}-\text{after}} \tag{3}$$

$$m_1v_{1\text{b}}+m_2v_{2\text{b}}=m_1v_{1\text{a}}+m_2v_{2\text{a}} \tag{3s}$$

*Definition of the Coefficient of Restitution*

The kinematic Coefficient of Restitution (*CoR*) was defined by Sir Isaac Newton as the ratio of the relative velocity of the two objects after the collision to the relative velocity before the collision.

In our models, for a collision at the sweet spot (ss) of the bat we have

$$CoR_{2\text{b}}=-\frac{v_{\text{ball}-\text{after}}-v_{\text{bat}-\text{cm}-\text{after}}-d_{\text{cm}-\text{ss}}\omega_{\text{bat}-\text{after}}}{v_{\text{ball}-\text{before}}-v_{\text{bat}-\text{cm}-\text{before}}-d_{\text{cm}-\text{ss}}\omega_{\text{bat}-\text{before}}} \tag{4}$$

$$CoR_{2\text{b}}=-\frac{v_{1\text{a}}-v_{2\text{a}}-d\omega_{2a}}{v_{1\text{b}}-v_{2\text{b}}-d\omega_{2\text{b}}} \tag{4s}$$

These variables are illustrated in Fig. 2. A note on notation: $\omega_{\text{bat}-\text{after}}$ is the angular velocity of the bat *about its center of mass* after the collision and

$v_{\text{bat}-\text{cm}-\text{before}}$ is the linear velocity *of the center of mass* of the bat in the x-direction before the collision: this is a combination of translation and rotation.

*Newton's Second Principle*

Watts and Bahill (1990) derived the following equation from Newton's second principle that states that a force acting on an object produces acceleration in accordance with the equation $F = ma$. If an object is accelerating, then its velocity and momentum is increasing. This principle is often stated as; applying an impulsive force to an object will change its momentum. According to Newton's third principle, when a ball hits a bat at the sweet spot there will be a force on the bat in the direction of the negative x-axis, let us call this $-F_1$, and an equal but opposite force on the ball, called $F_1$. This force will be applied during the duration of the collision. When a force is applied for a short period of time, it is called an impulse. According to Newton's second principle, an impulse will change momentum. The force on the bat will create a torque of $-d_{cm-ss}F_1$ around the center of mass of the bat. An impulsive torque will produce a change in angular momentum of the *bat*.

$$-d_{cm-ss}F_1 t_c = I_{bat}(\omega_{\text{bat}-\text{after}} - \omega_{\text{bat}-\text{before}})$$

Now this impulse will also change the linear momentum of the *ball*.

$$F_1 t_c = m_{\text{ball}}(v_{\text{ball}-\text{after}} - v_{\text{ball}-\text{before}})$$

Multiply both sides of this equation by $d_{\text{cm}-\text{ss}}$ and add these two equations to get

$$d_{\text{cm}-\text{ss}} m_{\text{ball}}(v_{\text{ball}-\text{after}} - v_{\text{ball}-\text{before}}) = -I_{\text{bat}}(\omega_{\text{bat}-\text{after}} - \omega_{\text{bat}-\text{before}}) \tag{5}$$

$$dm_1(v_{1\text{a}} - v_{1\text{b}}) = -I_2(\omega_{2\text{a}} - \omega_{2\text{b}}) \tag{5s}$$

For now, we have ignored $\omega_{\text{ball}}$. We will reconsider this later.

*Conservation of Angular Momentum*

The initial and final angular momenta comprise ball translation, ball rotation, bat translation and bat rotation about its center of mass.

$$L_{\text{initial}} = L_{\text{final}}$$

$$\begin{aligned} m_1 v_{1\text{b}} d + (I_1 + m_1 d^2)\omega_{1b} + I_2\omega_{2\text{b}} \\ = +m_1 v_{1\text{a}} d + (I_1 + m_1 d^2)\omega_{1\text{a}} + I_2\omega_{2\text{a}} \end{aligned} \tag{6s}$$

Summary of abbreviations that will be used in the following sections, with units:

$$C = v_{1b} - v_{2b} - d\omega_{2b} \quad \text{m/s}$$
$$D = \frac{m_1 d^2}{I_2} \quad \text{unit less}$$
$$K = (m_1 I_2 + m_2 I_2 + m_1 m_2 d^2) \quad \text{kg}^2\,\text{m}^2$$
$$\text{L} = v_{2b} m_2 I_2 (1 + CoR_{2b}) + \omega_{2b} m_2 d I_2 (1 + CoR_{2b}) \quad \text{kg}^2\,\text{m}^3/\text{s}$$
$$\bar{m} = \frac{m_1 m_2}{m_1 + m_2} \quad \text{kg}$$

Note that none of these abbreviations contains the outputs $v_{\text{ball-after}}$, $\omega_{\text{ball-after}}$, $v_{\text{bat-cm-after}}$, $\omega_{\text{bat-after}}$ and $KE_{\text{lost}}$. The most useful abbreviations are the ones that are constants independent of velocities after the collision. These abbreviations are only used during the derivations. They are removed from the output equations. We will now use the Newtonian principles in Eqs. (3)–(5) to find $v_{\text{ball-after}}$, $v_{\text{bat-cm-after}}$, and $\omega_{\text{bat-after}}$.

## Finding Ball Velocity After the Collision

First, we solve for $v_{\text{ball-after}}$.

Start with Eq. (5) and solve for $\omega_{2a}$

$$dm_1(v_{1a} - v_{1b}) = -I_2(\omega_{2a} - \omega_{2b})$$

$$\boxed{\omega_{2a} = \omega_{2b} - \frac{dm_1}{I_2}(v_{1a} - v_{1b})}$$

This equation was derived from Eq. (5). We will use this expression repeatedly. We know that for baseball and softball $\omega_{2b}$ is close to zero, but for generality, we will leave it in for as long as we can.

Next, we use Eq. (4) and solve for $v_{2a}$

$$CoR_{2b} = -\frac{v_{1a} - v_{2a} - d\omega_{2a}}{v_{1b} - v_{2b} - d\omega_{2b}}$$
$$CoR_{2b}(v_{1b} - v_{2b} - d\omega_{2b}) = -v_{1a} + v_{2a} + d\omega_{2a}$$
$$\boxed{v_{2a} = v_{1a} + CoR_{2b}(v_{1b} - v_{2b} - d\omega_{2b}) - d\omega_{2a}}$$

This equation was derived from Eq. (4). We will use this expression repeatedly. Next, substitute $\omega_{2a}$ into this $v_{2a}$ equation. We put substitutions in squiggly braces { } to make it obvious what has been inserted.

$$v_{2a} = v_{1a} + CoR_{2b}\ (v_{1b} - v_{2b} - d\omega_{2b}) - d\left\{\omega_{2b} - \frac{dm_1}{I_2}(v_{1a} - v_{1b})\right\}$$

Let $D = \frac{m_1 d^2}{I_2}$ and $C = v_{1b} - v_{2b} - d\omega_{2b}$

$$v_{2a} = v_{1a} + \{D\}(v_{1a} - v_{1b}) + CoR_{2b}\{C\} - d\omega_{2b}$$
$$v_{2a} = v_{1a}(1 + D) - v_{1b}D + CoR_{2b}C\ - d\omega_{2b}$$

Now substitute this $m_2 v_{2a}$ into Eq. (3)

$$m_1 v_{1b} + m_2 v_{2b} = m_1 v_{1a} + \{m_2 v_{1a}(1 + D) - m_2 D v_{1b} + m_2 CoR_{2b}\ C - m_2 d\omega_{2b}\}$$

Replace the dummy variables C and D and

$$v_{1a}\left[m_1 + m_2 + \frac{m_1 m_2 d^2}{I_2}\right] = v_{1b}\left[m_1 + \frac{m_1 m_2 d^2}{I_2} - m_2 CoR_{2b}\right] + m_2 v_{2b}$$
$$+ m_2 CoR_{2b} v_{2b}\ + \omega_{2b} m_2 d(1 + CoR_{2b})$$

Multiply by $I_2$.

$$v_{1a}\left[m_1 I_2 + m_2 I_2 + m_1 m_2 d^2\right] = v_{1b}\left[m_1 I_2 + m_1 m_2 d^2 - m_2 CoR_{2b} I_2\right] + m_2 v_{2b} I_2$$
$$+ m_2 CoR v_{2b}\ I_2 + \omega_{2b} m_2 d I_2 (1 + CoR_{2b})$$

Rearrange

$$\boxed{v_{1a} = \frac{v_{1b}(m_1 I_2 - m_2 I_2\ CoR_{2b} + m_1 m_2 d^2) + v_{2b} m_2 I_2 (1 + CoR_{2b}) + \omega_{2b} m_2 d I_2 (1 + CoR_{2b})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}} \tag{7}$$

This equation was derived from Eqs. (3)–(5).
Now we want to rearrange this normal form equation into its canonical form.

$$\text{Let } K = \left(m_1 I_2 + m_2 I_2 + m_1 m_2 d^2\right)$$
$$\mathrm{L} = v_{2b} m_2 I_2 (1 + CoR_{2b}) + \omega_{2b} m_2 d I_2 (1 + CoR_{2b})$$
$$v_{1a} = \frac{v_{1b}(m_1 I_2 - m_2 I_2 CoR_{2b} + m_1 m_2 d^2)}{K} + \frac{L}{K}$$

add $\left(v_{1b} - \frac{v_{1b} K}{K}\right)$ to the right side

$$v_{1a} = v_{1b} + \frac{v_{1b}(m_1 I_2 - m_2 I_2 CoR_{2b} + m_1 m_2 d^2) - v_{1b}(m_1 I_2 + m_2 I_2 + m_1 m_2 d^2)}{K} + \frac{L}{K}$$

$$v_{1a} = v_{1b} + \frac{-v_{1b} m_2 I_2 (1 + CoR_{2b}) + L}{K}$$

Finally, we get the canonical form:

$$\boxed{v_{1a} = v_{1b} - \frac{(v_{1b} - v_{2b}) m_2 I_2 (1 + CoR_{2b}) - \omega_{2b} m_2 d I_2 (1 + CoR_{2b})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}} \tag{8}$$

This equation was derived from Eqs. (3)–(5).

$$v_{\text{ball-after}} = v_{\text{ball-before}} - \frac{(v_{\text{ball-before}} - v_{\text{bat-cm-before}}) m_{\text{bat}} I_{\text{bat}} (1 + CoR_{2b}) - \omega_{bat-before} m_{bat} d I_{bat} (1 + CoR_{2b})}{m_{\text{ball}} I_{\text{bat}} + m_{\text{bat}} I_{\text{bat}} + m_{\text{ball}} m_{\text{bat}} d^2_{\text{cm-ss}}}$$

## Finding Bat Velocity After the Collision

We solve for $v_{\text{ball-after}}$. As before, we start with Eq. (5) and solve for $\omega_{2a}$

$$\omega_{2a} = \omega_{2b} - \frac{d m_1}{I_2}(v_{1a} - v_{1b})$$

Next use Eq. (4) and solve for $v_{2a}$

$$CoR_{2b} = -\frac{v_{1a} - v_{2a} - d\omega_{2a}}{v_{1b} - v_{2b} - d\omega_{2b}}$$

$$v_{2a} = v_{1a} + CoR_{2b}(v_{1b} - v_{2b} - d\omega_{2b}) - d\omega_{2a}$$

We will use this expression repeatedly. Substitute $\omega_{2a}$ into this $v_{2a}$ equation. I put the substitution in squiggly braces { } to make it obvious what has been inserted.

$$v_{2a} = v_{1a} + CoR_{2b}\ (v_{1b} - v_{2b} - d\omega_{2b}) - d\left\{\omega_{2b} - \frac{d m_1}{I_2}(v_{1a} - v_{1b})\right\}$$

Let $C = v_{1b} - v_{2b} - d\omega_{2b}$

$$v_{2a} = v_{1a} + \frac{m_1 d^2}{I_2}(v_{1a} - v_{1b}) + CoR_{2b}\{C\} - \omega_{2b} d$$

Equation (7) in the previous section is

$$v_{1a} = \left\{ v_{1b} - \frac{(v_{1b} - v_{2b})m_2 I_2(1 + CoR_{2b}) - \omega_{2b} m_2 d I_2(1 + CoR_{2b})}{K} \right\}$$

Put this into both places for $v_{1a}$ in the $v_{2a}$ equation above.

$$\begin{aligned} v_{2a} = & \left\{ v_{1b} - \frac{(v_{1b} - v_{2b})m_2 I_2(1 + CoR_{2b}) - \omega_{2b} m_2 d I_2(1 + CoR_{2b})}{K} \right\} \\ & + \frac{m_1 d^2}{I_2} \left( \left\{ v_{1b} - \frac{(v_{1b} - v_{2b})m_2 I_2(1 + CoR_{2b}) - \omega_{2b} m_2 d I_2(1 + CoR_{2b})}{K} \right\} - v_{1b} \right) \\ & + CoR_{2b} C - \omega_{2b} d \end{aligned}$$

Now multiply by K

$$\begin{aligned} v_{2a} K = & v_{1b} K - (v_{1b} - v_{2b}) m_2 I_2 (1 + CoR_{2b}) + \omega_{2b} m_2 d I_2 (1 + CoR_{2b}) \\ & + \frac{m_1 d^2}{I_2} \left[ v_{1b} K - (v_{1b} - v_{2b}) m_2 I_2 (1 + CoR_{2b}) + \omega_{2b} m_2 d I_2 (1 + CoR_{2b}) - v_{1b} K \right] \\ & + CoR_{2b}\, CK - \omega_{2b} d K \end{aligned}$$

Cancel the terms in color

$$\begin{aligned} v_{2a} K = & v_{1b} K - (v_{1b} - v_{2b}) m_2 I_2 (1 + CoR_{2b}) + \omega_{2b} m_2 d I_2 (1 + CoR_{2b}) \\ & - (v_{1b} - v_{2b}) m_1 m_2 d^2 (1 + CoR_{2b}) + \omega_{2b} m_1 m_2 d^3 (1 + CoR_{2b}) \\ & + CCoR_{2b}\, K - \omega_{2b} d K \end{aligned}$$

Let us break up the $(v_{1b} - v_{2b})$ terms and substitute $C = v_{1b} - v_{2b} - d\omega_{2b}$.

$$\begin{aligned} v_{2a} K = & v_{1b} K - v_{1b} m_2 I_2 (1 + CoR_{2b}) + v_{2b} m_2 I_2 (1 + CoR_{2b}) + \omega_{2b} m_2 d I_2 (1 + CoR_{2b}) \\ & - v_{1b} m_1 m_2 d^2 (1 + CoR_{2b}) + v_{2b} m_1 m_2 d^2 (1 + CoR_{2b}) + \omega_{2b} m_1 m_2 d^3 (1 + CoR_{2b}) \\ & + v_{1b} CoR_{2b}\, K - 2\mathrm{b} v CoR_{2b}\, K - \omega_{2b} d K (1 + CoR) \end{aligned}$$

Rearrange

$$\begin{aligned} v_{2a} K = & v_{1b} K - v_{1b} m_2 I_2 (1 + CoR_{2b}) - v_{1b} m_1 m_2 d^2 (1 + CoR_{2b}) + v_{1b} CoR_{2b}\, K \\ & + v_{2b} m_2 I_2 (1 + CoR_{2b}) + v_{2b} m_1 m_2 d^2 (1 + CoR_{2b}) - v_{2b} CoR_{2b}\, K \\ & + \omega_{2b} m_2 d I_2 (1 + CoR_{2b}) + \omega_{2b} m_1 m_2 d^3 (1 + CoR_{2b}) - \omega_{2b} d K (1 + CoR) \end{aligned}$$

Now let us break up the $(1 + CoR_{2b})$ terms.

$$\begin{aligned} v_{2a} K = & v_{1b} K - v_{1b} m_2 I_2 - v_{1b} m_2 I_2 CoR_{2b} - v_{1b} m_1 m_2 d^2 - v_{1b} m_1 m_2 d^2 CoR_{2b} + v_{1b} CoR_{2b}\, K \\ & + v_{2b} m_2 I_2 + v_{2b} m_2 I CoR_{2b} + v_{2b} m_1 m_2 d^2 + v_{2b} m_1 m_2 d^2 CoR_{2b} - v_{2b} CoR_{2b} K \\ & + \omega_{2b} m_2 d I_2 + \omega_{2b} m_2 d I_2 CoR_{2b} + \omega_{2b} m_1 m_2 d^3 + \omega_{2b} m_1 m_2 d^3 CoR_{2b} - \omega_{2b} d K - \omega_{2b} d K CoR_{2b} \end{aligned}$$

Are any of these terms the same? No. OK, now let's substitute

$K = (m_1 I_2 + m_2 I_2 + m_1 m_2 d^2)$ and hope for cancellations.

$$
\begin{aligned}
v_{2a}K = {} & v_{1b}\left(m_1 I_2 + m_2 I_2 + m_1 m_2 d^2\right) - v_{1b} m_2 I_2 - v_{1b} m_2 I_2 CoR_{2b} \\
& - v_{1b} m_1 m_2 d^2 - v_{1b} m_1 m_2 d^2 CoR_{2b} + v_{1b} CoR_{2b}\left(m_1 I_2 + m_2 I_2 + m_1 m_2 d^2\right) \\
& + v_{2b} m_2 I_2 + v_{2b} m_2 I_2 CoR_{2b} + v_{2b} m_1 m_2 d^2 + v_{2b} m_1 m_2 d^2 CoR_{2b} \\
& - v_{2b} CoR_{2b}\left(m_1 I_2 + m_2 I_2 + m_1 m_2 d^2\right) \\
& + \omega_{2b} m_2 d I_2 + \omega_{2b} m_2 d I_2 CoR_{2b} + \omega_{2b} m_1 m_2 d^3 + \omega_{2b} m_1 m_2 d^3 CoR_{2b} \\
& - \omega_{2b}\left(m_1 d I_2 + m_2 d I_2 + m_1 m_2 d^3\right) - \omega_{2b}\left(m_1 d I_2 + m_2 d I_2 + m_1 m_2 d^3\right) CoR_{2b}
\end{aligned}
$$

The terms in color cancel, leaving

$$v_{2a}K = v_{1b} m_1 I_2 (1 + CoR_{2b}) + v_{2b}(-m_1 I_2 CoR_{2b} + m_2 I_2 + m_1 m_2 d^2) - \omega_{2b} m_1 d I_2 (1 + CoR_{2b})$$

Continuing

$$v_{2a}K = +v_{1b} m_1 I_2 (1 + CoR_{2b}) + v_{2b}\left(-m_1 I_2 CoR_{2b} + m_2 I_2 + m_1 m_2 d^2\right) - \omega_{2b} m_1 d I_2 (1 + CoR_{2b})$$

distribute the second term and add $-v_{2b} m_1 I_2 + v_{2b} m_1 I_2$

$$
\begin{aligned}
v_{2a}K &= +v_{1b} m_1 I_2 (1 + CoR_{2b}) - v_{2b} m_1 I_2 CoR_{2b} - v_{2b} m_1 I_2 + v_{2b} m_1 I_2 + v_{2b} m_2 I_2 + v_{2b} m_1 m_2 d^2 - \omega_{2b} m_1 d I_2 (1 + CoR_{2b}) \\
&= \left(v_{1b} - v_{2b}\right) m_1 I_2 (1 + CoR_{2b}) + v_{2b} m_1 I_2 + v_{2b} m_2 I_2 + v_{2b} m_1 m_2 d^2 - \omega_{2b} m_1 d I_2 (1 + CoR_{2b}) \\
&= (v_{1b} - v_{2b}) m_1 I_2 (1 + CoR_{2b}) + v_{2b} K - \omega_{2b} m_1 d I_2 (1 + CoR_{2b}) \\
&= v_{2b} K + (v_{1b} - v_{2b}) m_1 I_2 (1 + CoR_{2b}) - \omega_{2b} m_1 d I_2 (1 + CoR_{2b})
\end{aligned}
$$

Finally divide by $K$

$$\boxed{v_{2a} = v_{2b} + \frac{(v_{1b} - v_{2b}) m_1 I_2 (1 + CoR_{2b}) - \omega_{2b} m_1 d I_2 (1 + CoR_{2b})}{(m_1 I_2 + m_2 I_2 + m_1 m_2 d^2)}}$$

This equation was derived from Eqs. (3)–(5) and (7). We can change this into our normal form by first combining the two terms over one common denominator.

$$
\begin{aligned}
v_{2a} &= v_{2b} \frac{(m_1 I_2 + m_2 I_2 + m_1 m_2 d^2)}{(m_1 I_2 + m_2 I_2 + m_1 m_2 d^2)} + \frac{(v_{1b} - v_{2b}) m_1 I_2 (1 + CoR_{2b}) - \omega_{2b} m_1 d I_2 (1 + CoR_{2b})}{(m_1 I_2 + m_2 I_2 + m_1 m_2 d^2)} \\
&= \frac{v_{2b}(m_1 I_2 + m_2 I_2 + m_1 m_2 d^2) + (v_{1b} - v_{2b}) m_1 I_2 (1 + CoR_{2b}) - \omega_{2b} m_1 d I_2 (1 + CoR_{2b})}{(m_1 I_2 + m_2 I_2 + m_1 m_2 d^2)}
\end{aligned}
$$

and then simplifying

$$\boxed{v_{2a} = \frac{v_{2b}(-m_1 I_2 CoR_{2b} + m_2 I_2 + m_1 m_2 d^2) + v_{1b} m_1 I_2 (1 + CoR_{2b}) - \omega_{2b} m_1 d I_2 (1 + CoR_{2b})}{(m_1 I_2 + m_2 I_2 + m_1 m_2 d^2)}}$$

or we can write this more compactly as

$$v_{2a}K = v_{2b}\left(-m_1 I_2 CoR_{2b} + m_2 I_2 + m_1 m_2 d^2\right) + v_{1b} m_1 I_2 (1 + CoR_{2b}) - \omega_{2b} m_1 d I_2 (1 + CoR_{2b})$$

## Finding the Bat Angular Velocity After the Collision

Now we want to find $\omega_{2a}$ (the angular velocity of the bat after the collision) in terms of the input parameters. We know that $\omega_{2b}$ is about zero, but for generality, we will leave it in for now.

This is $v_{1a}$ from the canonical form of Eq. (7).

$$v_{1a} = \left\{ v_{1b} - \frac{(v_{1b} - v_{2b}) m_2 I_2 (1 + CoR_{2b}) - \omega_{2b} m_2 d I_2 (1 + CoR_{2b})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2} \right\}$$

From Eq. (5) solve for $\omega_{2a}$

$$\omega_{2a} = \omega_{2b} - \frac{m_1 d}{I_2}(v_{1a} - v_{1b})$$

Substitute $v_{1a}$ into this equation for $\omega_{2a}$

$$\omega_{2a} = \omega_{2b} - \frac{m_1 d}{I_2}\left\{ v_{1b} - \frac{(v_{1b} - v_{2b}) m_2 I_2 (1 + CoR_{2b}) - m_2 d \omega_{2b} I_2 (1 + CoR_{2b})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2} \right\} + \frac{m_1 d}{I_2} v_{1b}$$

Finally

$$\boxed{\omega_{2a} = \omega_{2b} + \frac{(v_{1b} - v_{2b}) m_1 m_2 d (1 + CoR) - \omega_{2b} m_1 m_2 d^2 (1 + CoR_{2b})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}}$$

This equation was derived from Eqs. (5) and (7). We can change this into our normal form by first combining the two terms over one common denominator.

$$\begin{aligned}\omega_{2a} &= \omega_{2b} \frac{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2} + \frac{(v_{1b} - v_{2b}) m_1 m_2 d (1 + CoR) - m_1 m_2 d^2 \omega_{2b} (1 + CoR_{2b})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2} \\ &= \frac{\omega_{2b}\left(m_1 I_2 + m_2 I_2 + m_1 m_2 d^2\right) + (v_{1b} - v_{2b}) m_1 m_2 d (1 + CoR) - m_1 m_2 d^2 \omega_{2b} (1 + CoR_{2b})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}\end{aligned}$$

Cancel duplicate terms and we get the normal form

$$\boxed{\omega_{2a} = \frac{\omega_{2b}(m_1 I_2 + m_2 I_2 - m_1 m_2 d^2 CoR_{2b}) + (v_{1b} - v_{2b}) m_1 m_2 d(1 + CoR_{2b})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}}$$

Three Output Equations in Three Formats

We will now summarize by giving equations for $v_{\text{ball-after}}$, $v_{\text{bat-cm-after}}$ and $\omega_{\text{bat-after}}$ in three formats. First normal form

$$v_{1a} = \frac{v_{1b}(m_1 I_2 - m_2 I_2\ CoR_{2b} + m_1 m_2 d^2) + v_{2b} m_2 I_2 (1 + CoR_{2b}) + \omega_{2b} m_2 d I_2 (1 + CoR_{2b})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}$$

$$v_{2a} = \frac{v_{2b}(-m_1 I_2 CoR_{2b} + m_2 I_2 + m_1 m_2 d^2) + v_{1b} m_1 I_2 (1 + CoR_{2b}) - \omega_{2b} d m_1 I_2 (1 + CoR_{2b})}{(m_1 I_2 + m_2 I_2 + m_1 m_2 d^2)}$$

$$\omega_{2a} = \frac{\omega_{2b}(m_1 I_2 + m_2 I_2 - m_1 m_2 d^2 CoR_{2b}) + (v_{1b} - v_{2b}) m_1 m_2 d(1 + CoR_{2b})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}$$

Second canonical form

$$v_{1a} = v_{1b} - \frac{(v_{1b} - v_{2b}) m_2 I_2 (1 + CoR_{2b}) - \omega_{2b} m_2 d I_2 (1 + CoR_{2b})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}$$

$$v_{2a} = v_{2b} + \frac{(v_{1b} - v_{2b}) m_1 I_2 (1 + CoR_{2b}) - \omega_{2b} d m_1 I_2 (1 + CoR_{2b})}{(m_1 I_2 + m_2 I_2 + m_1 m_2 d^2)}$$

$$\omega_{2a} = \omega_{2b} + \frac{(v_{1b} - v_{2b}) m_1 m_2 d(1 + CoR_{2b}) - \omega_{2b} m_1 m_2 d^2 (1 + CoR_{2b})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}$$

Now let

$$A = \left\{ \frac{[(v_{1b} - v_{2b}) - \omega_{2b} d](1 + CoR_{2b})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2} \right\}$$

and we get our reduced canonical form:

$$v_{1a} = v_{1b} - A m_2 I_2$$
$$v_{2a} = v_{2b} + A m_1 I_2$$
$$\omega_{2a} = \omega_{2b} + A m_1 m_2 d$$

Please note that $A$ is not a constant. It depends on the inputs $v_{1b}, v_{2b}$ and $\omega_{2b}$. Also, notice that $\omega_{\text{ball}}$ does not appear in these output equations. It will appear later. We now want to add the equation for conservation of energy, Eq. (2).

Adding Conservation of Energy and Finding $KE_{\text{lost}}$

This approach, of adding conservation of energy to the bat-ball collision equations, is unique in the science of baseball literature. For a head-on collision at the center of mass of the bat, we had that

$$KE_{\text{lost}-\text{config}-\text{cm}} = \frac{\bar{m}}{2}\left(v_{\text{bat}-\text{cm}-\text{before}} - v_{\text{ball}-\text{before}}\right)^2\left(1 - CoR_{1\text{b}}^2\right) \tag{9}$$

However, for a collision at the sweet spot this equation for kinetic energy lost is not valid, because we now also have angular kinetic energy in the rotation of the bat. There are no springs in the system and the bat swing is level, therefore there is no change in potential energy. Before the collision, there is kinetic energy in the bat created by rotation of the batter's body and arms plus the translational kinetic energy of the ball. In Fig. 2, the sweet spot is the distance $d_{\text{cm}-\text{ss}}$ from the center of mass.

$$KE_{\text{before}} = \frac{1}{2}m_{\text{ball}}v_{\text{ball}-\text{before}}^2 + \frac{1}{2}m_{\text{bat}}v_{\text{bat}-\text{cm}-\text{before}}^2 + \frac{1}{2}I_{\text{ball}}\omega_{\text{ball}-\text{before}}^2 + \frac{1}{2}I_{\text{bat}}\omega_{\text{bat}-\text{before}}^2$$

As always, $\omega$ means rotation about the center of mass of the object. The collision will make the bat spin about its center of mass. If the collision is at the Center of Percussion for the pivot point, it will produce a rotation about the center of mass, but no translation.

$$KE_{\text{after}} = \frac{1}{2}m_{\text{ball}}v_{\text{ball}-\text{after}}^2 + \frac{1}{2}m_{\text{bat}}v_{\text{bat}-\text{cm}-\text{after}}^2 + \frac{1}{2}I_{\text{ball}}\omega_{\text{ball}-\text{after}}^2 + \frac{1}{2}I_{\text{bat}}\omega_{\text{bat}-\text{after}}^2$$

We now add kinetic energy of the rotating curveball. We will add two terms with ball spin $\left(\frac{1}{2}I_{\text{ball}}\omega_{\text{ball}-\text{before}}^2 \text{ and } \frac{1}{2}I_{\text{ball}}\omega_{\text{ball}-\text{after}}^2\right)$ to the Conservation of Energy equation, to create

$$\begin{aligned}
&\frac{1}{2}m_{\text{ball}}v_{\text{ball}-\text{before}}^2 + \frac{1}{2}m_{\text{bat}}v_{\text{bat}-\text{cm}-\text{before}}^2 + \frac{1}{2}I_{\text{ball}}\omega_{\text{ball}-\text{before}}^2 + \frac{1}{2}I_{\text{bat}}\omega_{\text{bat}-\text{before}}^2 \\
&\quad = \frac{1}{2}m_{\text{ball}}v_{\text{ball}-\text{after}}^2 + \frac{1}{2}m_{\text{bat}}v_{\text{bat}-\text{cm}-\text{after}}^2 + \frac{1}{2}I_{\text{ball}}\omega_{\text{ball}-\text{after}}^2 + \frac{1}{2}I_{\text{bat}}\omega_{\text{bat}-\text{after}}^2 + KE_{lost} \\
&KE_{\text{before}} = KE_{\text{after}} + KE_{\text{lost}}
\end{aligned}$$

The $KE_{\text{before}}$ and the $KE_{\text{after}} >$ are easy to find. It is the $KE_{\text{lost}}$ that is hard to find.

In the next section on "Adding Conservation of Angular Momentum," we will prove that for head-on collisions without friction $\omega_{ball-before} = \omega_{ball-after}$. Therefore, the ball spin terms in these conservation of energy equations cancel resulting in

$$0 = m_1 v_{1\text{b}}^2 + m_2 v_{2\text{b}}^2 + I_2\omega_{2\text{b}}^2 - m_1 v_{1\text{a}}^2 - m_2 v_{2\text{a}}^2 - I_2\omega_{2\text{a}}^2 - 2KE_{\text{lost}}$$

From before, we have

$$A = \left[\frac{(v_{1b} - v_{2b})(1 + CoR_{2b}) - d\omega_{2b}}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}\right]$$
$$v_{1a} = v_{1b} - Am_2 I_2$$
$$v_{1a} = v_{1b} - Am_2 I_2$$
$$v_{2a} = v_{2b} + Am_1 I_2$$
$$\omega_{2a} = \omega_{2b} + Am_1 m_2 d$$
$$\omega_{1a} = \omega_{1b}$$

Substituting $v_{1a}$, $v_{2a}$ and $\omega_{2a}$ into the new conservation of energy equation yields

$$KE_{\text{lost}} = \frac{1}{2}\left\{\begin{array}{l} m_1 v_{1b}^2 + m_2 v_{2b}^2 + I_2 \omega_{2b}^2 - m_1 (v_{1b} - Am_2 I_2)^2 \\ -m_2 (v_{2b} + Am_1 I_2)^2 - I_2 (\omega_{2b} + Am_1 m_2 d)^2 \end{array}\right\}$$

Now we want to put this into the form that we had for Eq. (1). The following derivation is original. First, expand the squared terms.

$$\begin{aligned} 2KE_{\text{lost}} &= m_1 v_{1b}^2 + m_2 v_{2b}^2 + I_2 \omega_{2b}^2 - m_1 (v_{1b}^2 - 2v_{1b} Am_2 I_2 + A^2 m_2^2 I_2^2) \\ &\quad -m_2 (v_{2b}^2 + 2v_{2b} Am_1 I_2 + A^2 m_1^2 I_2^2) - I_2 \left(\omega_{2b}^2 + 2\omega_{2b} Am_1 m_2 d + A^2 m_1^2 m_2^2 d^2\right) \\ &\quad \text{cancel terms in the same color and distribute the leading term} \\ 2KE_{\text{lost}} &= 2v_{1b} Am_1 m_2 I_2 - A^2 m_1 m_2^2 I_2^2 \\ &\quad -2v_{2b} Am_1 m_2 I_2 - A^2 m_1^2 m_2 I_2^2 - I_2 \left(2\omega_{2b} Am_1 m_2 d + A^2 m_1^2 m_2^2 d^2\right) \end{aligned}$$

Rearrange

$$2KE_{\text{lost}} = 2v_{1b} Am_1 m_2 I_2 - 2v_{2b} Am_1 m_2 I_2 - A^2 m_1^2 m_2 I_2^2 - A^2 m_1 m_2^2 I_2^2 - \left(2\omega_{2b} Am_1 m_2 d + A^2 m_1^2 m_2^2 d^2\right) I_2$$

factor

$$2KE_{\text{lost}} = Am_1 m_2 I_2 \left[2(v_{1b} - v_{2b}) - A(m_1 I_2 + m_2 I_2 + m_1 m_2 d^2) - 2\omega_{2b} d\right]$$

Substitute $A$

$$2KE_{\text{lost}} = Am_1 m_2 I_2 \left[2(v_{1b} - v_{2b}) - \left\{\frac{(v_{1b} - v_{2b})(1 + CoR_{2b}) - d\omega_{2b}}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}\right\}(m_1 I_2 + m_2 I_2 + m_1 m_2 d^2) - 2\omega_{2b} d\right]$$
$$2KE_{\text{lost}} = Am_1 m_2 I_2 [2(v_{1b} - v_{2b}) - (v_{1b} - v_{2b})(1 + CoR_{2b}) + d\omega_{2b} - 2\omega_{2b} d]$$

factor $(v_{1b} - v_{2b})$ out of the first two terms

$$2KE_{\text{lost}} = Am_1m_2I_2[(v_{1b} - v_{2b})(1 + CoR_{2b}) - d\omega_{2b}]$$

substitute A

$$2KE_{\text{lost}} = \left\{\frac{(v_{1b} - v_{2b})(1 + CoR_{2b}) - d\omega_{2b}}{m_1I_2 + m_2I_2 + m_1m_2d^2}\right\} m_1m_2I_2[(v_{1b} - v_{2b})(1 + CoR_{2b}) - d\omega_{2b}]$$

$$2KE_{\text{lost}} = \frac{m_1m_2I_2}{m_1I_2 + m_2I_2 + m_1m_2d^2}\{(v_{1b} - v_{2b})(1 + CoR_{2b}) - d\omega_{2b}\}[(v_{1b} - v_{2b})(1 + CoR_{2b}) - d\omega_{2b}]$$

After a little bit of algebra we get

$$2KE_{\text{lost}} = \frac{m_1m_2I_2}{m_1I_2 + m_2I_2 + m_1m_2d^2}\left[(v_{1b} - v_{2b})^2(1 - CoR_{2b}^2) - 2(v_{1b} - v_{2b})\omega_{2b}d + \omega_{2b}^2d^2\right]$$

$$\boxed{KE_{\text{lost}} = \frac{1}{2}\frac{m_1m_2I_2}{m_1I_2 + m_2I_2 + m_1m_2d^2}\left[(v_{1b} - v_{2b})^2(1 - CoR_{2b}^2) - 2(v_{1b} - v_{2b})\omega_{2b}d + \omega_{2b}^2d^2\right]}$$

This is a general result. It is original and unique. For a collision at the center of mass, d $=0$. Therefore,

$$KE_{\text{lost}} = \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}(v_{1b} - v_{2b})^2(1 - CoR^2)$$

When we substitute, $\bar{m} = \frac{m_1m_2}{m_1 + m_2}$ we get

$$KE_{\text{lost}} = \frac{\bar{m}}{2}(v_{1b} - v_{2b})^2(1 - CoR^2) \tag{10}$$

Which is the same as the following equation that has been derived in the literature.

$$KE_{\text{lost}} = \frac{\bar{m}}{2}(v_{\text{bat-cm-before}} - v_{\text{ball-before}})^2(1 - CoR^2)$$

## Adding Conservation of Angular Momentum

In this section, we will prove that for a head-on collision without considering friction for a pitch of any spin there will be no change in the spin of the ball. To do this we will use the law of conservation of angular momentum about the center of mass of the bat. When the ball contacts the bat, as shown in Fig. 3, the ball has linear momentum of $m_{\text{ball}}v_{\text{ball-before}}$. However, the ball does not know if it is

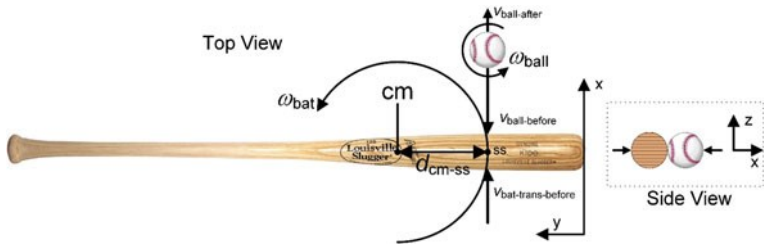


**Fig. 3** This figure shows $v_{\text{ball}-\text{before}}$, $v_{\text{ball}-\text{before}}$, $\omega_{\text{ball}}$, $d_{\text{cm}-\text{ss}}$ and $\omega_{\text{bat}}$, which are used in the conservation of angular momentum equation

translating or if it is tied on a string and rotating about the center of mass of the bat. Following conventional physics, we will model the ball as rotating about the bat's center of mass at a distance $d = d_{\text{cm}-\text{ss}}$. Therefore, the ball has an initial angular momentum of $m_{\text{ball}} d_{\text{cm}-\text{ss}} v_{\text{ball}-\text{before}}$ about the bat's center of mass. In addition, it is possible to throw a curveball so that it spins about the vertical, z-axis, as also shown in Fig. 3. We call this a purely horizontal curveball (although it will still drop due to gravity, more than it will curve horizontally). The curveball will have angular momentum of $I_{\text{ball}} \omega_{\text{ball}-\text{before}}$ where $I_{\text{ball}} = 0.4 m_{\text{ball}} r_{\text{ball}}^2$ about an axis parallel to the z-axis. However, this is its momentum about *its* center of mass and we want the momentum about the center of mass of the *bat*. Therefore, we use the parallel axis theorem producing $(I_{\text{ball}} + m_{ball} d^2)\omega_{\text{ball}-\text{before}}$.

The bat has an initial angular momentum of $I_{\text{bat}} \omega_{\text{bat}-\text{before}}$. It also has an angular momentum about the bat's center of mass of due to the bat translation momentum $m_{\text{bat}} d v_{\text{bat}-\text{before}}$, however, in this case $d = 0$ because the center of mass of the bat is passing through its center of mass. $L$ is the symbol used for angular momentum. I guess all the cool letters (like $F$, $m$, $a$, $v$, $I$, $\omega$, $d$, etc.) were already taken, so they were stuck with the blah symbol $L$. Therefore, the initial angular momentum about the center of mass of the bat is

$$L_{\text{initial}} = m_1 v_{1\text{b}} d + \left(I_1 + m_1 d^2\right)\omega_{1b} + I_2 \omega_{2\text{b}}$$

All of these momenta are positive, pointing out of the page.

For the final angular momentum, we will treat the ball, as before, as an object rotating around the axis of the center of mass of the bat with angular momentum, $m_{\text{ball}} v_{\text{ball}-\text{after}} d_{\text{cm}-\text{ss}}$. Now we could treat the bat as a long slender rod with a moment of inertia of $m_{\text{bat}} d_{\text{bat}}^2/12$, where $d_{\text{bat}}$ is the bat length. However, this is only an approximation and we have actual experimental data for the bat moment of inertia. Therefore, the bat angular momentum is $I_{\text{bat}} \omega_{\text{bat}-\text{after}}$. Thus, our final angular momentum about the center of mass of the bat is

$$L_{\text{final}} = m_1 v_{1\text{a}} d + \left(I_1 + m_1 d^2\right)\omega_{1\text{a}} + I_2 \omega_{2\text{a}}$$

The law of conservation of angular momentum states that the initial angular momentum about some axis equals the final angular momentum about that axis.

$$\boxed{\begin{array}{l} L_{\text{initial}} = L_{\text{final}} \\ m_1 v_{1\text{b}} d + (I_1 + m_1 d^2)\omega_{1b} + I_2\omega_{2\text{b}} = m_1 v_{1\text{a}} d + (I_1 + m_1 d^2)\omega_{1\text{a}} + I_2\omega_{2\text{a}} \end{array}}$$

Previously we used Eq. (5), Newton's second principle and solved for $\omega_{2\text{a}}$.

$$dm_1(v_{1\text{a}} - v_{1\text{b}}) = -I_2(\omega_{2\text{a}} - \omega_{2\text{b}}) \tag{11}$$

$$\omega_{2\text{a}} = \omega_{2\text{b}} - \frac{dm_1}{I_2}(v_{1\text{a}} - v_{1\text{b}})$$

So let us substitute this into our conservation of angular momentum equation above.

$$m_1 v_{1\text{b}} d + I_1\omega_{1b} + m_1\omega_{1b}d^2 + I_2\omega_{2b} = m_1 v_{1\text{a}} d + I_1\omega_{1a} + m_1\omega_{1a}d^2 + I_2\left\{\omega_{2\text{b}} + \frac{dm_1}{I_2}(v_{1\text{b}} - v_{1\text{a}})\right\}$$

We want to solve this for $\omega_{1\text{a}}$

$$-I_1\omega_{1\text{a}} - m_1\omega_{1\text{a}}d^2 = -m_1 v_{1\text{b}} d - I_1\omega_{1b} - I_2\omega_{2b} - m_1\omega_{1b}d^2 + m_1 v_{1\text{a}} d + I_2\omega_{2\text{b}} + dm_1(v_{1\text{b}} - v_{1\text{a}})$$

Cancel the terms in color and rearrange

$$\omega_{1\text{a}}(I_1 + m_1 d^2) = \omega_{1b}(I_1 + m_1 d^2)$$

$$\boxed{\omega_{1\text{a}} = \omega_{1b}}$$

We have now proven that for a pitch with any spin about the z-axis, the spins before and after are the same. What about a pitch that has spin about the z-axis and also about the y-axis, like most pitches? The collision will not change ball rotation. As shown above, it will not change the spin about the z-axis. We could write another set of equations for angular momentum about the y-axis. However, the bat has no angular momentum about the y-axis, so there is nothing to affect the ball spin about the y-axis. In conclusion, a head-on collision between a bat and a ball will not change the spin on the ball. Some papers have shown a relationship between ball spin before and ball spin after, but they were using oblique collisions (Nathan et al. 2012; Kensrud et al. 2016) (Table 3).

The numbers in the Excel simulation satisfy the following checks: (1) Conservation of linear momentum, (2) Conservation of angular momentum, (3) Coefficient of restitution, (4) Newton's second principle, an impulse changes momentum, (5) Conservation of energy and (6) Kinetic energy lost. Table 4 shows the kinetic energies for the same simulation.

The first purpose of this paper is to model bat-ball collisions using only Newton's principles and the conservation equations. We did it. Our equations are complete, consistent and correct.

**Table 3** Simulation values for bat-ball collisions at the sweet spot

| | SI units (m/s, rad/s, or J) | Baseball units |
|---|---|---|
| *Inputs* | | |
| $v_{\text{ball-before}}$ | −37 | −83 mph |
| $\omega_{\text{ball-before}}$ | 209 | 2000 rpm |
| $v_{\text{bat-cm-before}}$ | 26 | 58 mph |
| $\omega_{\text{bat-before}}$ | 0.1 | 1 rpm |
| $CoR_{2b}$ | 0.55 | |
| *Outputs* | | |
| $v_{\text{ball-after}}$ | 43 | 97 mph |
| $\omega_{\text{ball-after}}$ | $=\omega_{\text{ball-before}}$ | |
| $v_{\text{bat-cm-after}}$ | 13 | 29 mph |
| $\omega_{\text{bat-after}}$ | −32 | −310 rpm |
| $KE_{\text{lost}}$ | 165 | |

**Table 4** Kinetic energies

| | |
|---|---|
| KE ball linear velocity before | 99.3 |
| KE bat linear velocity before | 304.2 |
| KE ball angular velocity before | 1.7 |
| KE bat angular velocity before | 0.0 |
| KE before total | 405.2 |
| KE ball linear velocity after | 136.1 |
| KE bat linear velocity after | 77.2 |
| KE ball angular velocity after | 1.7 |
| KE bat angular velocity after | 25.2 |
| KE after | 240.2 |
| KE loss | 165.0 |
| KE after + KE loss | 405.2 |

### 3.2.5 Analytic Sensitivity Analysis

The second purpose of this paper is to show how the batter can select and tailor an optimal baseball or softball bat. From the viewpoint of the batter, the only model output that is important is the speed of the batted-ball. Therefore, we will now find the sensitivity of the batted-ball speed, $v_{\text{ball-after}}$, with respect to the system parameters. The eight system parameters are $v_{\text{ball-before}}$, $m_{\text{ball}}$, $I_{\text{bat}}$, $m_{\text{bat}}$, $CoR_{2b}$, $d_{\text{cm-ss}}$, $v_{\text{bat-cm-before}}$ and $\omega_{\text{bat-before}}$. For baseball and softball, the batted – ball speed, $v_{1a}$, is the most important output. The larger it is the more likely the batter will get on base safely (Baldwin and Bahill 2004). Therefore, let us start with $v_{1a}$ from Eq. (7).

$$v_{1a} = v_{1b} - \frac{(v_{1b} - v_{2b})m_2 I_2(1 + CoR_{2b}) - \omega_{2b} m_2 d I_2(1 + CoR_{2b})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}$$

In order to perform an analytic sensitivity analysis we first need the partial derivatives of $v_{1a}$ with respect to the eight parameters. These partial derivatives are often called the absolute sensitivity functions. Move the minus sign and simplify the numerator.

$$v_{1a} = v_{1b} + \frac{(1 + CoR_{2b})[(-v_{1b} + v_{2b})m_2 I_2 + \omega_{2b} m_2 d I_2]}{(m_1 I_2 + m_2 I_2 + m_1 m_2 d^2)}$$

Let $K = (m_1 I_2 + m_2 I_2 + m_1 m_2 d^2)$

$$H = (1 + CoR_{2b})[(-v_{1b} + v_{2b})m_2 I_2 + \omega_{2b} m_2 d I_2]$$

$$v_{1a} = v_{1b} + \frac{H}{K}$$

$$\frac{\partial v_{1a}}{\partial v_{1b}} = 1 - \frac{m_2 I_2(1 + CoR_{2b})}{K} \quad \text{unitless}$$

$$\frac{\partial v_{1a}}{\partial \omega_{2b}} = \frac{m_2 d I_2(1 + CoR_{2b})}{K} \quad \text{m}$$

$$\frac{\partial v_{1a}}{\partial CoR_{2b}} = \frac{(-v_{1b} + v_{2b})m_2 I_2 + \omega_{2b} m_2 d I_2}{K} \quad \text{m/s}$$

$$\frac{\partial v_{1a}}{\partial v_{2b}} = \frac{m_2 I_2(1 + CoR_{2b})}{K} \quad \text{unitless}$$

Alternatively, we could start with

$$v_{1a} = v_{1b} - A m_2 I_2$$

$$A = \left\{ \frac{[(v_{1b} - v_{2b}) - \omega_{2b} d](1 + CoR_{2b})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2} \right\}$$

$$v_{1a} = v_{1b} - \left\{ \frac{[(v_{1b} - v_{2b}) - \omega_{2b} d](1 + CoR_{2b})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2} \right\} m_2 I_2$$

$$\frac{\partial v_{1a}}{\partial v_{2b}} = \frac{m_2 I_2(1 + CoR_{2b})}{K}$$

This gives the same result. For the following partial derivatives, we need the derivative of a quotient.

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x)f'f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$\frac{\partial v_{1a}}{\partial d} = \frac{(1 + CoR_{2b})K m_2 \omega_{2b} I_2 - 2H m_1 m_2 d}{K^2} \quad 1/\text{s}$$

$$\frac{\partial v_{1a}}{\partial m_2} = \frac{K(1+CoR_{2b})\{(-v_{1b}+v_{2b})I_2+\omega_{2b}dI_2\}-H(I_2+m_1d^2)}{K^2} \quad \text{m/kg s}$$

$$\frac{\partial v_{1a}}{\partial m_1} = -\frac{(I_2+m_2d^2)H}{K^2} \quad \text{m/kg s}$$

$$\frac{\partial v_{1a}}{\partial I_2} = \frac{K(1+CoR_{2b})[(-v_{1b}+v_{2b})m_2+\omega_{2b}m_2d]-H(m_1+m_2)}{K^2} \quad \text{1/kg m s}$$

$$\frac{\partial^2 v_{1a}}{\partial v_{2b}\partial m_2} = \frac{I_2(1+CoR_{2b})[K-m_2(I_2+m_1d^2)]}{K^2} \quad \text{1/kg}$$

In the above partial derivatives, units on the left and right sides of the equations are the same. This is a simple, but important accuracy check. We perform such a dimensional analysis on all of our equations.

We did not show the derivations of all of the second-order partial derivatives. We choose the interaction of the bat mass and the bat speed, above, because it was expected to be large based on principles of physiology. Additionally, the forthcoming discussion on optimizing the bat suggests an interaction between the bat mass and moment of inertia. Therefore, we will now derive one more interaction term, the interaction between bat mass and moment of inertia, $I_{bat}$ and $m_{bat}$.

Given

$$\frac{\partial v_{1a}}{\partial m_2} = \frac{K(1+CoR_{2b})\{(-v_{1b}+v_{2b})I_2+\omega_{2b}dI_2\}-H(I_2+m_1d^2)}{K^2}$$

Find $\frac{\partial^2 v_{1a}}{\partial I_2 \partial m_2}$

We will be dealing with $I_2$, so let us isolate it. First replace K and H, $\frac{\partial v_{1a}}{\partial m_2}$ becomes

$$= \left[\left(m_1+m_2\right)I_2+m_1m_2d^2\right]\left(1+CoR_{2b}\right)\left(-v_{1b}+v_{2b}+\omega_{2b}d\right)I_2$$
$$-\left(1+CoR_{2b}\right)\left(-v_{1b}+v_{2b}+\omega_{2b}d\right)m_2I_2\left(I_2+m_1d^2\right)$$
$$= \left(1+CoR_{2b}\right)\left(-v_{1b}+v_{2b}+\omega_{2b}d\right)(m_1I_2^2+m_2I_2^2+m_1m_2d^2I_2-m_2I_2^2-m_1m_2d^2I_2)$$

Cancel the terms in color and consolodate the terms without $I_2$ by letting

$$E = \left(1+CoR_{2b}\right)\left(-v_{1b}+v_{2b}+\omega_{2b}d\right)$$

The numerator $\frac{\partial v_{1a}}{\partial m_2}$ of becomes

$$= Em_1 I_2^2$$

Therefore,

$$\frac{\partial^2 v_{1a}}{\partial I_2 \partial m_2} = \frac{2Em_1 I_2 K^2 - Em_1 I_2^2 2K\left(m_1 + m_2\right)}{K^4}$$

$$= \frac{2Em_1 I_2 K\left[K - I_2\left(m_1 + m_2\right)\right]}{K^4}$$

substitute for the second $K$ in the numerator

$$= \frac{2Em_1 I_2 K\left[\left\{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2\right\} - I_2\left(m_1 + m_2\right)\right]}{K^4}$$

$$= \frac{2Em_1 I_2 K\left[m_1 I_2 + m_2 I_2 + m_1 m_2 d^2 - m_1 I_2 - m_2 I_2\right]}{K^4}$$

cancel the terms in color

$$= \frac{2Em_1^2 m_2 d^2 I_2 K}{K^4}$$

substitute $E$ and $K$

$$\frac{\partial^2 v_{1a}}{\partial I_2 \partial m_2} = \frac{2\left\{\left(1 + CoR_{2b}\right)\left(-v_{1b} + v_{2b} + \omega_{2b} d\right)\right\} m_1^2 m_2 d^2 I_2}{\left\{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2\right\}^3} \quad 1/\mathrm{kg}^2 \bullet \mathrm{m} \bullet \mathrm{s}$$

Now we want to form the *semirelative-sensitivity functions,* which are defined as

$$\tilde{S}_\alpha^F = \left.\frac{\partial F}{\partial \alpha}\right|_{\mathrm{NOP}} \alpha_0$$

where NOP and the subscript 0 mean that all functions, inputs and parameters assume their nominal operating point values (Smith et al. 2008).

$$\tilde{S}_\alpha^F = \left.\frac{\partial F}{\partial \alpha}\right|_{\mathrm{NOP}} \alpha_0$$

$$\tilde{S}_{v_{1b}}^{v_{1a}} = 1 - \left.\frac{m_2 I_2 (1 + CoR_{2b})}{K}\right|_{\mathrm{NOP}} v_{1b_0}$$

$$\tilde{S}_{v_{2b}}^{v_{1a}} = \left.\frac{m_2 I_2 (1 + CoR_{2b})}{K}\right|_{\mathrm{NOP}} v_{2b_0}$$

$$\tilde{S}_{\omega_{2b}}^{v_{1a}} = \left.\frac{m_2 d I_2 (1 + CoR_{2b})}{K}\right|_{\mathrm{NOP}} \omega_{2b_0}$$

$$\tilde{S}_{CoR}^{v_{1a}} = \left.\frac{(-v_{1b} + v_{2b}) m_2 I_2 + \omega_{2b} m_2 d I_2}{K}\right|_{\mathrm{NOP}} CoR_{2b_0}$$

**Table 5** Typical values and sensitivities

| Variable | Nominal values | | Range of realistic values | | $\tilde{S}_{\alpha}^{F} = \frac{\partial F}{\partial \alpha}\Big\vert_{\mathrm{NOP}} \alpha_0$ semirelative sensitivity values |
|---|---|---|---|---|---|
| | SI units | Baseball units | SI units | Baseball units | |
| *Inputs* | | | | | |
| $v_{\mathrm{ball-before}}$ | −37 m/s | −83 mph | 26.8–40.2 m/s | 60–90 mph | 10 |
| $v_{\mathrm{bat-cm-before}}$ | 26 m/s | 58 mph | 24.6–27.2 m/s | 58 ± 10 mph | 31 |
| $\omega_{\mathrm{ball-before}}$ | 209 rad/s | 2000 rpm | 188–230 rad/s | 2000 ± 100 rpm | 0 |
| $\omega_{\mathrm{bat-before}}$ | 0.1 rad/s | 1 rpm | −0.1–0.1 rad/s | ±2 rpm | 0.02 |
| *Parameters* | | | | | |
| $CoR_{2\mathrm{b}}$ | 0.55 | | 0.45–0.65 | 0.55 ± 0.1 | 28 |
| $d_{\mathrm{cm-ss}}$ | 0.134 m | 5.3 in. | 0.13–0.14 m | 5.3 ± 2 in. | −7 |
| $m_{\mathrm{ball}}$ | 0.145 kg | 5.125 oz | 0.142–0.156 kg | 5.125 ± 0.125 oz | −14 |
| $m_{\mathrm{bat}}$ | 0.905 kg | 32 oz | 0.709–0.964 kg | 25–34 oz | 10 |
| $I_{\mathrm{bat-cm}}$ | 0.048 kg m$^2$ | 2624 oz in$^2$ | 0.036–0.06 kg m$^2$ | 1968–3280 oz in$^2$ | 4 |
| $v_{\mathrm{bat-before}}$ interacting with $m_{\mathrm{bat}}$ | | | | | 4 |
| $I_{\mathrm{bat}}$ interacting with $m_{\mathrm{bat}}$ | | | | | 1 |

$$\tilde{S}_{d}^{v_{1a}} = \frac{(1+CoR_{2b})K\omega_{2b}m_2I_2 - 2Hm_1m_2d}{K^2}\bigg|_{\text{NOP}} d_0$$

$$\tilde{S}_{m_2}^{v_{1a}} = \frac{K(1+CoR_{2b})[(-v_{1b}+v_{2b})I_2 + d\omega_{2b}I_2] - H(I_2+m_1d^2)}{K^2}\bigg|_{\text{NOP}} m_{2_0}$$

$$\tilde{S}_{m_1}^{v_{1a}} = -\frac{(I_2+m_2d^2)H}{K^2}\bigg|_{\text{NOP}} m_{1_0}$$

$$\tilde{S}_{I_2}^{v_{1a}} = \frac{K(1+CoR_{2b})[(-v_{1b}+v_{2b})m_2 + m_2d\omega_{2b}] - H(m_1+m_2)}{K^2}\bigg|_{\text{NOP}} I_{2_0}$$

$$\tilde{S}_{v_{2b}-m_2}^{v_{1a}} = \frac{I_2(1+CoR_{2b})[K - m_2(I_2+m_1d^2)]}{K^2}\bigg|_{\text{NOP}} v_{2b_0}m_{2_0}$$

$$\tilde{S}_{I_2-m_2}^{v_{1a}} = \frac{2[(1+CoR_{2b})(-v_{1b}+v_{2b}+\omega_{2b}d)]m_1^2m_2d^2I_2}{[m_1I_2+m_2I_2+m_1m_2d^2]^3}\bigg|_{\text{NOP}} I_{2_0}m_{2_0}$$

Table 5 gives the nominal input and parameter values, along with a range of physically realistic values for collegiate and professional batters and the semirelative sensitivity values. The bigger the sensitivity is, the more important the variable is for maximizing batted-ball speed.

The right column of Table 5 shows that the most important variable, in terms of maximizing batted-ball speed, is the speed of the bat before the collision. This is certainly no surprise. The second most important variable is the coefficient of restitution,$CoR_{2b}$. The least important variables are the angular velocities, $\omega_{\text{ball-before}}$ and $\omega_{\text{bat-before}}$. The sensitivities to distance between the center of mass and the sweet spot of the bat, $d_{\text{cm-ss}}$, and the mass of the ball, $m_{\text{ball}}$, are negative, which merely means that as they increase the batted-ball speed decreases. Cross (2011) wrote that in his model the most sensitive variables were also the bat speed followed by the *CoR*. His sensitivity to the mass of the ball was also negative. The bottom two rows of Table 5 show that the interaction terms are small, which means that the model is well behaved. For example, the interaction of the mass of the bat with the bat speed is smaller than either the influence of the mass of the bat by itself or the bat speed by itself. The interaction of bat mass and moment of inertia is surprisingly small.

### 3.2.6 Optimizing with Commercial Software

We applied *What's Best!,* a subset of the LINGO solvers, to our model. We constrained each variable to stay within physically realistic limits under natural conditions. Such values are shown in Table 5. We have previously gotten good results using this technique when doing empirical sensitivity analyses (Bahill et al. 2009). Then we asked the optimizer to give us the set of values that would maximize batted-ball speed. The optimizer applied a nonlinear optimization program. The

results were the same as in Table 5! That is, for variables with positive sensitivities, the optimizer choose the maximum values. For variables with negative sensitivities, the optimizer choose the minimum values. Using all of the optimal values at the same time increased the batted-ball speed from 43 to 56 m/s (96–125 mph). Using this optimal set of values only changed the sensitivities slightly.

1. The numerical sensitivity values mostly increased. This is a direct result of the definition of the semirelative sensitivity function where the partial derivative is multiplied by the parameter value. If parameter values increase, then the sensitivities increase.
2. However, and most importantly, the rank order stayed the same except that the batted-ball speed became more sensitive to $v_{\text{ball}-\text{before}}$ than to $m_{\text{bat}}$. In the optimal set, both of these sensitivities increased, but because the value of $v_{\text{ball}-\text{before}}$ changed from 37 to 40 m/s whereas the value of $m_{\text{bat}}$ only changed from 0.90 to 0.96 kg, the change in the sensitivity to $v_{\text{ball}-\text{before}}$ was bigger.

This all means that the sensitivity analysis is robust. Its results remain basically the same after big changes in the variables.

We then tried a different optimization technique. Instead of constraining each variable to stay within realistic physical limits, we allowed the optimizer to change each variable by at most ±10% and then give us the set of values that maximizes batted-ball speed. The numerical values changed but the rank order stayed the same, except for $v_{\text{ball}-\text{before}}$ and $m_{\text{bat}}$ just as it did with the realistic values technique.

Both empirical sensitivity analyses and optimization can constrain each variable to stay within specified realistic physical limits or change each variable by a certain percentage. Both techniques gave the same results. However, we prefer the former technique (Bahill et al. 2009).

We found an interesting relationship between the sensitivity analyses and optimization: they gave the same results! For variables with positive sensitivities, the optimizer chooses the maximum values. For variables with negative sensitivities, the optimizer chooses the minimum values. But of course, this finding is not original. Sensitivity analyses are commonly used in optimization studies (Choi and Kim 2005). These studies typically apply sensitivity analysis after optimization. They try to find values or limits for the objective function or the right-hand sides of the constraints that would change the decisions. However, in our study, we applied optimization after the sensitivity analysis and we had only one variable in our objective function. Therefore, our problem was much simpler than sensitivity analyses in the optimization literature.

### 3.2.7 Optimizing the Bat

The second purpose of this paper is to help the batter acquire an optimal baseball or softball bat. Therefore, we ask, How can the batter use these sensitivity and optimization results to select or customize an optimal bat? First, it is no surprise that bat speed, $v_{\text{bat}-\text{cm}-\text{before}}$, is the most important variable in Table 5. Its effect is shown

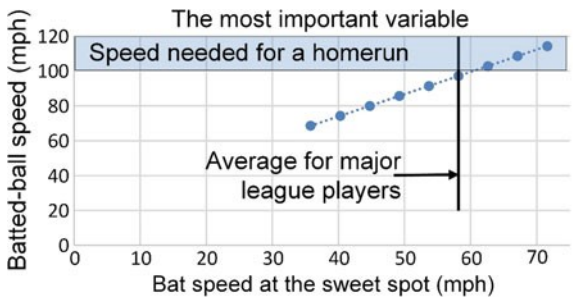


**Fig. 4** The most important variable in our model is the bat speed at the sweet spot before the collision. For this figure, first we computed the batted-ball speed with $v_{\text{ball-after}} = v_{\text{ball-before}} - Am_{\text{bat}}I_{\text{bat}}$ and then we plotted the batted-ball speed as a function of the bat sweet spot speed before the collision. Remember that $A$ is not a constant, it depends on the velocity of the ball and bat before the collision and on the angular velocity of the bat before the collision

in Fig. 4, where the slope of the line is the absolute sensitivity. For decades, Little League coaches have taught their boys to practice and gain strength so that they could increase their bat speeds. They also said that it is very important to reduce the variability in the bat swings: Every swing should be the same. "Don't try to kill the ball." Given our new information, we now recommend that Little League coaches continue to give the same advice: increase bat speed and reduce variability. Practice is the key. Baldwin (2007), a major-league pitcher with a career 3.08 ERA, sagaciously wrote that if you lose a game, don't blame the umpire or your teammates; just go home and practice harder.

Our measurements of over 300 batters showed that variability in the speed of the swing decreases with level of performance from Little League to Major League Baseball. For major leaguers the bat speed standard deviations were typically around ±5% (Bahill and Karnavas 1989), which is a very small value for physiological data.

The variable with the second largest sensitivity is the coefficient of restitution (*CoR*). The *CoR* of a bat-ball collision depends on where the ball hits the bat. It is difficult, but absolutely essential, for the batter to control this. He or she must hit the ball with the sweet spot of the bat. The *CoR* also depends on the manufacturing process. The NCAA now measures the Bat-ball Coefficient of Restitution (BBCOR) for sample lots coming off the manufacturing line. Therefore, amateurs are all going to get similar BBCORs. However, a lot can still be done with the *CoR* for aluminum and composite bats. For example, the performance of composite bats typically improves with age because of the break-in process; repeatedly striking the bat eventually breaks down the bat's composite fibers and resinous glues. 'Rolling' the bat also increases its flexibility. Rolling the bat stretches the composite fibers and accelerates the natural break-in process simulating a break-in period of hitting, say, 500 balls.

For wooden bats, the batter could try to influence the *CoR* by choosing the type of wood that the bat is made of. Throughout history, the most popular woods have been white ash, sugar maple and hickory. However, hickory is heavy, so most

professionals now use ash or maple. A new finding about bat manufacturing is that the slope of the grain has an effect on the strength and elasticity of the bat. As a result, the wood with the straightest gain is reserved for professionals and wood with the grain up to 5° off from the long-axis of the bat is used for amateurs. Furthermore, the manufacturer's emblem is stamped on the flat grain side of ash bats so that balls collide with edge grain as shown in Fig. 1, whereas the emblem is stamped on the edge grain side of maple bats because they are stronger when the collision is on the flat grain side.

The next largest sensitivities are for the mass of the ball and its speed before the collision, $m_{ball}$ and $v_{\mathrm{ball-before}}$. However, the batter can do nothing to influence the mass of the ball or the ball speed before the collision, so we will not concern ourselves with them. Likewise, the batter has no control over the ball spin, $\omega_{\mathrm{ball-before}}$, so we will ignore it when selecting bats. Now if this discussion were being written from the perspective of the pitcher (Baldwin 2007), then these three parameters would be very important.

The next most important variable in Table 5 is the mass of the bat. Therefore, we will now consider the mass and other related properties of the bat. The sensitivity of the batted-ball speed with respect to the mass of the bat is positive, meaning (if everything else is held constant) as the mass goes up so does the batted-ball speed. However, everything else cannot be held constant, because the heavier bat cannot be swung as fast (Bahill and Karnavas 1989) due to the force-velocity relationship of human muscle. This physiological relationship was not included in the equations of this paper because in this paper we only modeled the *physics* of the collision, notwithstanding physiology trumping physics in this case. The net result of physics *in conjunction with physiology* is that lighter bats are better for almost all batters (Bahill 2004).

Perhaps due to this general feeling, back in the 1960s and 70s, it was popular for professionals to 'cork' the bat. This reduced the mass of the bat, but because it also reduced the moment of inertia, it did not improve performance significantly (Nathan et al. 2011). However, it is now legal to make a one to two-inch diameter hole 1.25 in. deep into the barrel end of the bat. Most batters do this because it makes the bat lighter with few adverse effects. Other bat parameters that are being studied include the type of wood (density, strength, elasticity and straightness of the grain) and the type of materials (density, strength, break-in period and vibrational frequency).

For an aluminum bat, some batters reduce the thickness of the barrel wall by shaving the inside of the barrel. This reduces the bat mass, which according to physics *and* physiology, increases batted-ball speed.

The distance between the center of mass of the bat and the sweet spot, $d_{\mathrm{cm-ss}}$, is the next most important parameter. We presumed that the sweet spot of the bat was the center of percussion (CoP) of the bat. All batters try to hit the ball on the sweet spot of the bat. To help the batter, manufacturers of aluminum bats have been moving the CoP by moving the internal weight from the end of the bat toward the knob http://www.acs.psu.edu/drussell/bats/cop.html. It is now an annual game of cat and mouse. The manufacturers move the CoP, then the rule makers change their rules, then the manufacturers move … etc.

Finally, we come to the moment of inertia of the bat, $I_{bat}$, with respect to its center of mass. The physics, revealed with the sensitivity analysis, states that although the moment of inertia is one of the least important variables, it would help to increase its value. More importantly, physiology showed that all batters would profit from using end-loaded bats (Bahill 2004). There are many ways to change the moment of inertia of a bat. Most aluminum bats start with a common shell and then the manufacturer adds a weight inside to bring the bat up to its stated weight. The important question then becomes, *where* should the weight be added? It has been suggested that they add weight in the knob because this would comply with the regulations and would not decrease bat speed. However, the results of Bahill (2004) show that they should add the weight in the barrel end of the bat making it *end loaded*. This will increase the batted-ball speed. For a wooden bat, the moment of inertia can be changed by cupping out the barrel end, adding weight to the knob or tapering the barrel end. Assume that the end of the barrel of a bat is only used to "protect" the outside edge of the plate: no one hits home runs on the end of the bat. Therefore, a professional could use a bat where the last 3 in. (7 cm) was tapered from 2½ inches (6.4 cm) down to 1¾ of an inch (4.4 cm). This would decrease the weight, decrease the moment of inertia about the center of mass and would move the sweet spot 2% closer to the knob: these changes would probably benefit some players. However, such modifications would have to be individually designed for each player.

Most people can feel the difference between bats with different moments of inertia. In 1985, a coach with the San Francisco Giants showed us a legal custom-made bat with a large moment of inertia created by leaving it with a huge knob. He presumed that his players already understood the influence of bat weight on bat speed so he was trying to expand their understanding to the influence of bat moment of inertia on the speed of the swing. One of our University of Arizona softball players described our biggest moment of inertia bat, "That's the one that pulls your arms out."

The bat moment of inertia is the only parameter under the control of the batter for which a consensus does not exist in the science of baseball literature. The bat moment of inertia is an enigma because for most, but not all, batters as the bat moment of inertia goes up the *bat* speed goes down, and at the same time the *batted-ball* speed goes up (Bahill 2004; Smith and Kensrud 2014). For Bahill's (2004) batters, 20% had positive slopes for bat speed versus moment of inertia, for moments of inertia in the range of 0.03–0.09 kg m$^2$. Therefore, he showed the actual data for all players rather than averaging them, because averaging graphical data is usually meaningless. Perhaps more physiological studies would help clear up this issue. Our best generalization is that all batters would profit from using end-loaded bats. Smith and Kensrud (2014) concluded their paper with "Batter swing speed decreased with increasing bat inertia, while … the hit-ball speed increases with bat inertia."

Summarizing, these are the most important factors for understanding bat performance: bat weight, the coefficient of restitution, the moment of inertia and characteristics of humans swinging the bats.

In the future, it will be possible to see how the coefficient of friction $\mu_f$ affects the batted-ball speed. Then we will be able to decide if the varnish or paint on the bat

should be made rough-textured or smooth, or if bats should be rubbed or oiled in order to improve bat performance.

To improve bat performance manufacturers could reduce the variability of bat and ball parameters. Major-league bats were custom made for us by Hillerich and Bradsby Co. The manufacturing instructions were "Professional Baseball Bat, R161, Clear Lacquer, 34 in., 32 oz, make as close to exact as possible, end brand—genuine model R161 pro stock, watch weights" emphasis added. The result was six bats with an average weight of 32.1 oz and a standard deviation of 0.5! This large standard deviation surprised us. We assume there is the same variability in bats used by major-league players.

There is also variability in the ball. We assume that the center of mass of the ball is coincident with the geometric center of the ball. However, put a baseball or softball in a bowl of water. Let the movement subside. Then put an X on the top the ball. Now spin it and let the motion subside again. The X will be on top again. This shows that for most baseballs and softballs the center of mass is not coincident with the geometric center of the ball.

### 3.2.8 Summary of Bat Selection

These sensitivity and optimality analyses show that the most important variable, in terms of increasing batted-ball speed, is bat speed before the collision. This is in concert with ages of baseball folklore and principles of physiology. Therefore, batters should develop strength, increase coordination and practice so that their swings are fast and with low variability.

These analyses show that the next most important parameter is the coefficient of restitution, the *CoR*. Engineers and bat regulators are free to play their annual cat and mouse game of increasing *CoR* then writing rules and making tests that prohibit these changes. Indeed, most recent bat research has gone into increasing the *CoR* of bat-ball collisions.

Pitch speed, ball spin and the mass of the ball are important. However, the batter cannot control them. Therefore, they cannot help the batter to choose or modify a bat.

The next most important parameter is the bat mass, $m_{bat}$. However, physics recommends heavy bats, whereas the force-velocity relationship of muscle recommends light bats. In this case, physiology trumps physics. Each person's preferred bat should be as light as possible while still fitting within baseball needs, regulations and availability.

The last interesting parameter from the sensitivity analysis and the optimization study is the bat moment of inertia, $I_{bat}$. These studies suggest that a larger bat moment of inertia would be better. However, a lot of the physics literature recommends smaller moments of inertia. Conversely, an experimental physiology study stated that all players would profit from using end-loaded bats (Bahill 2004). Therefore, this is the only parameter under the control of the batter for which a consensus does not exist in the science of baseball literature.

The second purpose of this paper is to show what the batter can do to achieve optimal bat performance. The most important thing is practice. Next, batters should select lightweight bats. They should then select bats that increase the *CoR* by all legal means. Finally, they should choose bats with a larger moment of inertia, bats that are often called end-loaded.

### 3.2.9 The Ideal Bat Weight™

So far, the equations in this paper were equations of physics. However, we repeatedly mentioned physiology. Now is the time to step back and look at physiology. This section is based on Bahill and Karnavas (1991).

Our instrument for measuring bat speed, the[1] Bat Chooser™, has two vertical laser beams, each with associated light detectors. Our batters swung the bats through the laser beams. A computer recorded the time between interruptions of the light beams. Knowing the distance between the light beams and the time required for the bat to travel that distance, the computer calculated the speed of the sweet spot, which we defined as the center of percussion. We told the batters to swing each bat as fast as they could while still maintaining control. We said, "Pretend you are trying to hit a Nolan Ryan fastball."

In our experiments, each batter swung six bats through the light beams. The bats ran the gamut from super-light to super heavy; yet they had similar lengths and weight distributions. In our developmental experiments, we tried about four dozen bats. We used aluminum bats, wooden bats, plastic bats, heavy metal warm-up bats, bats with holes in them, bats with lead in them, major-league bats, college bats, softball bats, Little League bats, brand-new bats and bats made in the 1950s.

In one set of experiments, we used six bats of significantly different weights but similar lengths of about 34 in. (89 cm), with centers of mass about 23 in. from the end of the handle (see Table 6).

In a 20-min interval, each subject swung each bat through the instrument five times. The order of presentation was randomized. The selected bat was announced by a speech synthesizer, for example: "Please swing bat Hank Aaron, that is, bat A." (We named our bats after famous baseball players who had names starting with the letter assigned to the bat.)

For each swing, we recorded the bat weight and the speed of the center of mass, which we converted to the speed of the center of percussion. However, that was as far as physics could take us; we then had to look to the principles of physiology.

Physiologists have long known that muscle speed decreases with increasing load. This is why bicycles have gears; gears enable riders to maintain the muscle speed that imparts maximum power through the pedals, while the load, as reflected by the bicycle speed, varies greatly. To discover how the muscle properties of individual baseball players affect their best bat weights, for each player, we plotted

[1]Bat Chooser and Ideal Bat Weight are trademarks of Bahill Intelligent Computer Systems.

**Table 6** Test bats used by major-league players

| Name | Weight (oz) | Weight (kg) | Distance from knob to center of mass (in.) | Distance from knob to center of mass (m) | Average sweet spot speed (mph) from Fig. 5 | Description |
|---|---|---|---|---|---|---|
| D | 49.0 | 1.39 | 22.5 | 0.57 | 88 | Aluminum bat filled with water |
| C | 42.8 | 1.21 | 24.7 | 0.63 | 74 | Wooded bat, filled with lead |
| A | 33.0 | 0.94 | 23.6 | 0.60 | 65 | Wooded bat |
| B | 30.6 | 0.87 | 23.3 | 0.59 | 65 | Wooden bat |
| E | 23.6 | 0.67 | 23.6 | 0.60 | 61 | Wooden bat |
| F | 17.9 | 0.51 | 21.7 | 0.55 | 60 | Wooden handle mounted on a light steel pipe with a 6 oz weight at the end |

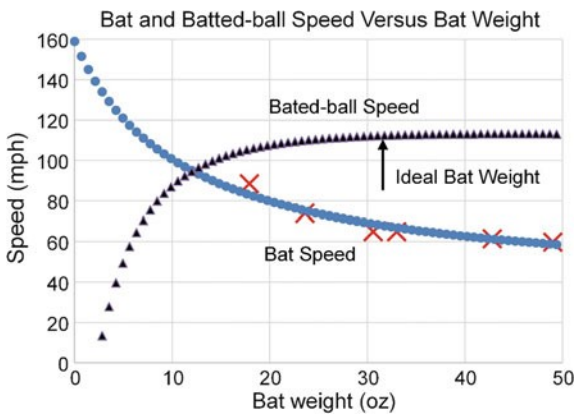


**Fig. 5** Measured bat speed (*red Xs*), a hyperbola fit to this data (*blue dots*) and the calculated batted-ball speed (*black triangles*) for a 90 mph pitch to one of the fastest San Francisco Giants

bat speeds as a function of bat weight to produce graphical numerical models known as the muscle force-velocity relationships (see Fig. 5). The red Xs represent the average of the five swings of each bat; the standard deviations were small for physiological data.

Over the past 75 years, physiologists have used three equations to describe the force-velocity relationship of muscles: straight lines, hyperbolas and exponentials. Each of these equations has produced the best fit for some experimenters, under certain conditions and with certain muscles. However, usually the hyperbola fits the data best. In our experiments, we tried all three equations and chose the one that had the best fit to the data of each subject's 30 swings. For the data of the force-velocity relationships illustrated in Fig. 5, we found that a hyperbola provided the best fit.

These curves indicate how bat speed varies with bat weight. We now want to find the bat weight that will make the ball leave the bat with the highest speed and thus have the greatest chance of eluding the fielders. We call this the maximum-batted-ball-speed bat weight. To calculate this bat weight we must couple the muscle force-velocity relationships to the equations of physics.

For the major-league player whose data are shown in Fig. 5, the best fit for his force-velocity data was the hyperbola, $(m_{\text{bat}} + 11) \times (v_{\text{bat-before}} - 36) = 1350$ units are ounces and mph, blue dots. This batter had some of the fastest swing speeds on the team. When we substituted this equation into the batted-ball speed equation, Eq. (7), we were able to plot the ball speed after the collision as a function of bat weight, black triangles in Fig. 5.

$$v_{1a} = \frac{v_{1b}(m_1 I_2 - m_2 I_2\ CoR_{2b} + m_1 m_2 d^2) + v_{2b} m_2 I_2 (1 + CoR_{2b}) + \omega_{2b} m_2 d I_2 (1 + CoR_{2b})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}$$

$$(m_{\text{bat}} + 11) \times (v_{2b} - 36) = 1350$$

$$v_{2b} = \left\{ \frac{36 m_2 + 1746}{m_2 + 11} \right\}$$

$$v_{1a} = v_{1b} \frac{(m_1 I_2 - m_2 I_2\ CoR_{2b} + m_1 m_2 d^2)}{K} + \left\{ \frac{36 m_2 + 1746}{m_2 + 11} \right\} \frac{m_2 I_2 (1 + CoR_{2b})}{K} + \omega_{2b} \frac{m_2 d I_2 (1 + CoR_{2b})}{K}$$

In this equation, $I_2$ is also a function of $m_2$. This curve shows that the maximum-batted-ball-speed bat weight for this subject is about 45 oz, which is much heavier than that used by any batters. However, this batted-ball speed curve is almost flat between 30 and 49 oz. This player normally used a 32-oz bat. Evidently the greater control permitted by the 32-oz bat outweighed the one per cent increase in speed that could be achieved with the 45-oz bat.

However, the maximum-batted-ball-speed bat weight is not the best bat weight for any player. Because a lighter bat will give a batter better control, more accuracy and more time to compute the ball's impact point. Obviously, a trade-off must be made between batted-ball speed and control. Because the batted-ball speed curve is so flat around the point of the maximum-batter-ball-speed, we believe there is little advantage in using a bat as heavy as the maximum-batter-ball-speed bat weight. Therefore, we have defined the [1]*ideal bat weight*™ to be the weight where the ball speed curve drops 1 per cent below the maximum-batter-ball speed. Using this criterion, the ideal bat weight for this batter is 31.75 oz. We believe this gives a good trade-off between distance and accuracy.

As can be seen from the batted-ball speed equation, both $v_{1a}$ and the ideal bat weight increase with pitch speed. However, we do not recommend that a batter use a heavier bat against a fire-baller, because heavier bats increase the swing time and decrease the prediction time.

The ideal bat weight is specific to each individual; it is not correlated with height, weight, age, circumference of the upper arm, or any combination of these factors, nor is it correlated with any other obvious physical factors. Although, Bahill

and Morna Freitas (1995) mined our database of 163 subjects and 36 factors and determined some rules of thumb that could make suggestions.

### 3.2.10 Bat Speed

Throughout this paper we have used a before collision bat speed of 58 mph (26 m/s). This is the average sweet spot speed that we measured for 28 members of the San Francisco Giants baseball team. However, our subjects were not paid and therefore they were not highly motivated: furthermore, they did not actually hit a ball: both of these circumstances increase the variance of swing speeds. Some studies in the literature filtered their data and only included selected batters, usually the fastest. Internet sites that are trying to sell their equipment and services cite sizzling bat speeds between 70 and 90 mph (31–40 m/s). We think that these numbers are bogus. The big web sites such as mlb.com, espn.com/mlb/and hittrackeronline.com give the leaders in many categories, meaning that they a have selected the 20 fastest players out of 750. This would be misleading if the reader thought that these statistics were *representative* of major-league batters, which they do.

Table 7 gives average sweet spot speeds for six studies of male college and professional batters. When multiple bats were used, we chose the wooden bats closest to that described in Table 1.

Figure 4 shows that the average major-league batter has a high enough bat speed to occasionally hit a home run, when the batted-ball has the ideal spin and launch angle. However, over half of major-league batters seldom hit homeruns. Indeed, of the 2200 active players listed by MLB.com half of them have never hit a home run in their major-league careers. Our equations show that a ball velocity before the collision, $v_{1b}$, of 83 mph (37 m/s) and a bat sweet spot speed, $v_{2b}$, of 58 mph

**Table 7** Bat sweet spot speed before the collision

| Average speed of the sweet spot (m/s) | Average speed of the sweet spot (mph) | Subjects | References |
|---|---|---|---|
| 32 | 71 | Unknown | King et al. (2012) |
| 31 | 69 | 7 selected male professional baseball players | Welch et al. (1995) |
| 30 | 68 | 19 male baseball players | Crisco et al. (2002) |
| 26 | 58 | 28 San Francisco Giants | Database of Bahill and Karnavas (1989) |
| 26 | 58 | 7 male college baseball players | Koenig et al. (2004) |
| 26 | 58 | 17 male college baseball players | Fleisig et al. (2002) |

(26 m/s) would produce a batted-ball speed, $v_{1a}$, of 97 mph (43 m/s), which would be almost enough for a home run in any major-league stadium. Our rule of thumb is that it takes a batted-ball speed of 100 mph (45 m/s) to produce a homerun. The following is Eq. (7).

$$v_{1a} = \frac{v_{1b}(m_1 I_2 - m_2 I_2\ CoR_{2b} + m_1 m_2 d^2) + v_{2b} m_2 I_2 (1 + CoR_{2b}) + \omega_{2b} m_2 d I_2 (1 + CoR_{2b})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}$$

For a major league wooden bat, as described in Table 1,

$$v_{1a} = -0.28 v_{1b} + 1.28 v_{2b} + 0.17 \omega_{2b}$$

where the units are either mph and rpm or m/s and rad/s. Remember that $v_{1b}$ is a negative number. So far, we have made no approximations; everything has been exactly according to Newton's principles. But now we will create our *rule of thumb* by rounding, substituting $\omega_{2b} = 0$ and using pitch speed instead the speed of the ball at the beginning of the collision.

$$\boxed{v_{\text{batted-ball}} = -0.25 v_{\text{pitch-speed}} + 1.3 v_{\text{bat-before}}}$$

For oblique collisions, the batted-ball speed would be less, but backspin on the ball in flight would keep it up in the air longer, so those two effects partially cancel out (Kensrud et al. 2016).

Most recent studies of bat speed have used video cameras and commercial prepackaged software to measure and compute bat speed. There are no calibration tests. Most of these systems report higher bat speeds than other methods of measuring bat speed. On television, the batted-ball speed is often called the exit speed or the exit velocity.

### 3.2.11 Seeing the Collision

When a baseball bat moving at 58 mph (26 m/s) hits a baseball traveling in the opposite direction at 83 mph (37 m/s) there is a violent collision, which was shown in figure 5.3. Table 5.3 shows that during the collision the kinetic energy in the motion of the bat changes by 81 Joules (J): a loss of 106 J in linear translational kinetic energy, a gain of 25 J in angular kinetic energy. Notably, 81 J is equivalent to dropping a bowling ball from your waist onto your toe or having a dove fly into your windshield while you are driving down a highway at 80 mph (130 km/hr).

Frame by frame analysis of high-speed video of a major-league batter showed that at the beginning of the collision there was (1) a big abrupt change in the ball velocity as it swung from negative to positive, (2) a sudden drop in the linear velocity of the sweet spot of the bat and (3) a sharp change in the angle of the bat.

Now, imagine a film of Ted Williams hitting a baseball. His swing is smooth and graceful although the kinetic energy of his bat changes by 202 Joules during a collision. The reason his swing seems so smooth is that we mainly visualize the movement of his body, arms, hands and the bat. We model this movement with the bat's angular rotation about the knob, β. The change in this angular motion is not visually obvious because it is just a short small jerk (a few degrees) in the middle of a big swinging motion. Hence, what we see does not change much. On the other hand, the bat's linear translational motion, β, decreases from 26 to 13 m/s. However, we do not visualize this translational motion well, because his swing looks like a big rotation: it does not look like a translation. As a result, the movement that we visualize well, does not change much. Whereas, the movement that changes a lot, β, is not visualized well. This explains why people do not perceive an abrupt jerk when the bat and ball collide.

What about the batter? Would he be able to see the effects of this violent collision? Probably not. Bahill and LaRitz (1984) showed that no batter can keep his eye on the ball from the pitcher's release point to the bat-ball collision. Their graduate students fell behind when the ball was 9 ft (2.7 m) in front of the plate. Comparatively, their major-league baseball player was able to keep his position error below 2° until the ball was 5.5 ft (1.7 m) from the plate. Then he fell behind. This finding runs contrary to baseball's hoary urban legend that Ted Williams could see the ball hit his bat. However, in reality, Ted Williams could not see the ball hit his bat. In a letter that he sent to Bahill dated January 23, 1984 he wrote,

> Received your letter and have also had a chance to read your research, and I fully agree with your findings.
>
> I always said I couldn't see a ball hit the bat except on very, very rare occasions and that was a slow pitch that I swung on at shoulder height. I cam[e] very close to seeing the ball hit the bat on those occasions.

In summary, the bat-ball collision is violent. But nobody perceives it, because (1) even in slow motion, the spectator only sees the smooth movement of the batters body, arms, hands, and bat, which glide continuously, (2) movements that change abruptly, such as the bat's linear translational velocity, are difficult to visualize because they are so quick, (3) batters are not able to see the bat-ball collision at all and (4) the bat-ball collision only lasts one millisecond. This explains why nobody sees an abrupt jerk when the bat hits the ball, not even Ted Williams.

## 4 Summary

One purpose of this paper was to show how complicated bat-ball collisions could be while still being modeled using only Newton's principles and the conservation laws. The model of this paper is the most complex configuration for which our model is valid. Our model was explained with Figs. 1 and 3. The five equations that we used were listed in Table 2.

The following canonical form equations comprise our model for bat-ball collisions.

$$KE_{\text{lost}} = \frac{1}{2}\frac{m_1 m_2 I_2}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}\left[(v_{1\text{b}} - v_{2\text{b}})^2(1 - CoR_{2\text{b}}^2) - 2(v_{1\text{b}} - v_{2\text{b}})\omega_{2\text{b}} d + d^2\omega_{2\text{b}}^2\right]$$

$$v_{1\text{a}} = v_{1\text{b}} - \frac{(v_{1\text{b}} - v_{2\text{b}})m_2 I_2(1 + CoR_{2\text{b}}) - \omega_{2\text{b}} m_2 d I_2(1 + CoR_{2\text{b}})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}$$

where $v_{2\text{b}} = v_{\text{bat-trans-before}} + d_{\text{cm-ss}}\omega_{\text{bat-before}}$

$$v_{2\text{a}} = v_{2\text{b}} + \frac{(v_{1\text{b}} - v_{2\text{b}})m_1 I_2(1 + CoR_{2\text{b}}) - \omega_{2\text{b}} d m_1 I_2(1 + CoR_{2\text{b}})}{(m_1 I_2 + m_2 I_2 + m_1 m_2 d^2)}$$

$$\omega_{2\text{a}} = \omega_{2\text{b}} + \frac{(v_{1\text{b}} - v_{2\text{b}})m_1 m_2 d(1 + CoR_{2\text{b}}) - \omega_{2\text{b}} m_1 m_2 d^2(1 + CoR_{2\text{b}})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}$$

$$\omega_{1a} = \omega_{1b}$$

If we let

$$A = \left\{\frac{[(v_{1\text{b}} - v_{2\text{b}}) - \omega_{2\text{b}} d](1 + CoR_{2\text{b}})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}\right\}$$

then we get

$$v_{1\text{a}} = v_{1\text{b}} - A m_2 I_2$$
$$v_{2\text{a}} = v_{2\text{b}} + A m_1 I_2$$
$$\omega_{2\text{a}} = \omega_{2\text{b}} + A m_1 m_2 d$$
$$\omega_{1a} = \omega_{1b}$$

A second purpose of this paper was to show how the individual batter can find and customize an optimal baseball or softball bat for him or herself. The sensitivity analysis and optimization study of this paper showed that the most important variable, in terms of increasing batted-ball speed, is bat speed before the collision. However, in today's world, the coefficient of restitution and the bat mass are experiencing the most experimentation trying to improve bat performance. Although, the bat moment of inertia provides more room for future improvement. Above all, future studies must include physics in conjunction with physiology in order to improve bat performance.

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