Sensitivity Analysis of Linear Homeomorphic Model for Human Movement

A. TERRY BAHILL, MEMBER, IEEE, JOSE R. LATIMER, AND B. TODD TROOST

Abstract—**Sensitivity functions were calculated as functions of time to help validate a linear sixth-order model for neuromuscular movements. For comparing parameters, relative sensitivity functions should be used. When the object of interest is a function of time, such as a system step response, semirelative sensitivity functions are best. Semirelative sensitivity functions were calculated for the 18 parameters of our model. The three most important parameters in the model were those describing the input controller signals. Because the sensitivity functions were functions of time, it**

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A. T. Bahill and J. R. Latimer are with the Biomedical Engineering Program, Department of Electrical Engineering, Carnegie-Mellon University, and the Department of Neurology, University of Pittsburgh Medical School, Pittsburgh, PA 15213.

B. T. Troost was with the Biomedical Engineering Program, Department of Electrical Engineering, Carnegie-Mellon University, and the Department of Neurology, University of Pittsburgh Medical School, Pittsburgh, PA. He is now with the Department of Neurology, Case Western Reserve Medical School, Case Western Reserve University, Cleveland, OH 44106.

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State Equations

$$\dot{x}_1 = \theta_1 = x_4$$

$$\dot{x}_2 = \dot{\theta}_2 = \frac{K_{SE}^2}{(K_{LT}+K_{SE})B_{AG}}x_1 - \frac{K_{SE}}{B_{AG}}x_2 + \frac{K_{SF}}{(K_{LT}+K_{SE})B_{AG}}x_5$$

$$\dot{x}_3 = \dot{\theta}_3 = \frac{K_{SE}^2}{(K_{LT}+K_{SE})B_{ANT}}x_1 - \frac{K_{SE}}{B_{ANT}}x_3 - \frac{K_{SE}}{(K_{LT}+K_{SE})B_{ANT}}x_6$$

$$\dot{x}_4 = \dot{\theta}_1 = \frac{-2K_{SE}-K_p}{J}x_1 + \frac{K_{SE}}{J}x_2 + \frac{K_{SE}}{J}x_3 - \frac{B_p}{J}x_4$$

$$\dot{x}_5 = F_{AG} = \frac{N_{AG}-x_5}{\tau_{AG}}$$

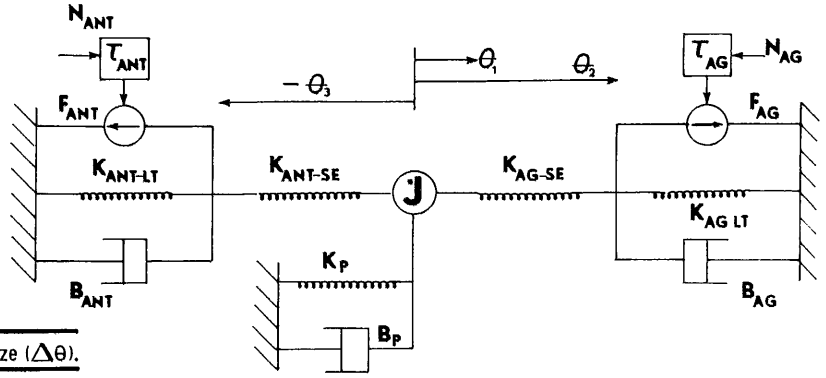
$$\dot{x}_6 = F_{ANT} = \frac{N_{ANT}-x_6}{\tau_{ANT}}$$

Initial Conditions

$x_1(0) = x_4(0) = 0$

$x_2(0) = -x_3(0) = 0.76$ mm

$x_5(0) = x_6(0) = 0.2$ newtons



Parameter Values as Functions of Saccade Size ($\Delta\theta$).

K_{SE} = 125 newtons/meter

K_{LT} = 60 n/m

K_p = 25 n/m

B_p = 3.1 n-s/m

B_{AG} = 2.36 n-s/m

B_{ANT} = 1.12 n-s/m

J = 2.2×10^{-3} n-s^2/m

$N_{AG\text{-}PULSE}$ = PH = $(0.54 + 0.11\Delta\theta)$n for $\Delta\theta \leq 11^0$

= $(1.57 + 0.02\Delta\theta)$ n for $\Delta\theta > 11^0$

$N_{ANT\text{-}PULSE}$ = 0.005 n

PW_{AG} = $(10 + \Delta\theta)$ ms

PW_{ANT} = PW_{AG} + 6 ms

$N_{AG\text{-}STEP}$ = $(0.2 + 0.02\Delta\theta)$ n

$N_{ANT\text{-}STEP}$ = $(0.2 - 0.007\Delta\theta)$ n

$t_{AG\text{-}AC}$ = $(11.7 - 0.2\Delta\theta)$ ms

$t_{AG\text{-}DE}$ = 0.2 ms

$t_{ANT\text{-}AC}$ = 2.4 ms

$t_{ANT\text{-}DE}$ = 1.9 ms

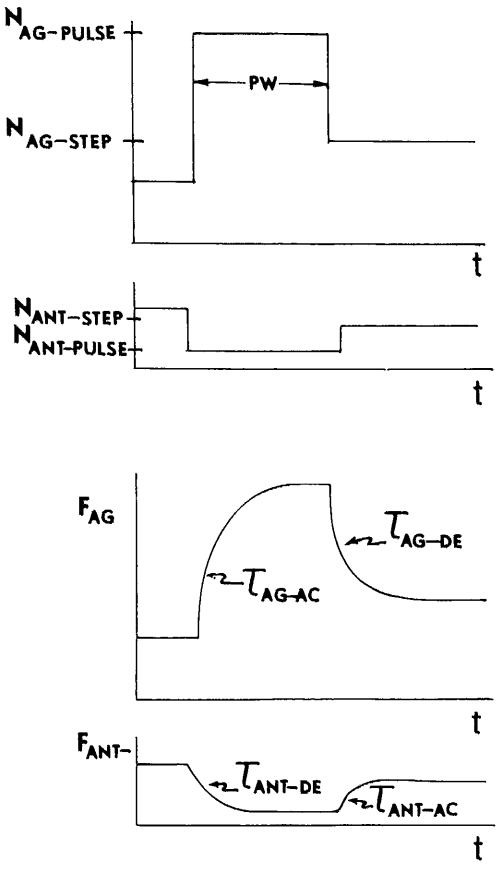


Fig. 1. Linear homeomorphic model for human eye movements. (See text for explanation.)

could be seen when each parameter had its greatest effect. This guided studies of individual parameters.

Introduction

Sensitivity analyses are an important method of validating economic models [1], societal models [2], engineering models [3], and physiological models [4], [5]. They show which parameters of the model have the most effect on the model behavior, and they allow a simplified treatment of unimportant parameters. If the sensitivity coefficients are calculated as functions of time, then it can be seen when each parameter has the greatest effect on the output function of interest. Sensitivity analyses results can be used to select numerical values for the parameters. The values of the parameters will be chosen to match the physiological data at the times when they have the most effect upon the output. Sensitivity analyses can also be used to suggest future experiments to elucidate the behavior of biological systems.

This correspondence describes a sensitivity analysis of a linear sixth-order model for human movement developed by Bahill *et al.* [6]. The parameter values were selected to match human eye movement values, but the form of the model should be applicable to all neuromuscular systems.

The Linear Homeomorphic Model

Fig. 1 is a complete description of the model. It shows parameter definitions, parameter values, controller signals, a model schematic, and the state equations. Saccadic eye movement control signals have two components: the pulse that moves the eye rapidly from one point to the next, and the step that holds the eye in the new position [7]. The pulse is composed of a high-frequency burst of motoneuronal firing for the agonist (AG) and a concomitant pulse of inhibition, a pause in motoneuronal firing, for the antagonist (ANT). Three model parameters are associated with the pulse: agonist pulsewidth (PW), agonist pulse height (PH), and antagonist pulse height ($N_{\text{ANT-pulse}}$). The step is composed of step changes in the steady state motoneuron firing rates: a step increase for the agonist and a step decrease for the antagonist. Two model parameters describe the step levels ($N_{\text{AG-step}}$ and $N_{\text{ANT-step}}$). The idealized pulse step controller signals are modified by first-order activation and deactivation time constants (the four τ's) to produce the muscle active state tensions (F_{AG} and F_{ANT}). The active state tensions are modified by the dashpots (B_{AG} and B_{ANT}), representing the muscle force–velocity relationships, the springs (K_{LT}), representing the muscle length-tension diagrams, and the springs (K_{SE}), representing the muscle series elasticities, to produce the muscle forces which are applied to the globe. The globe and surrounding tissues are modeled by the inertia (J), a viscous element (B_p) and a passive elasticity (K_p). The passive elasticities of each muscle are also incorporated into K_p. When no specific subscript is given, for example in the numerical values for the series elasticities, then the agonist and antagonist values are identical. The nominal values for the model parameters were selected to minimize the mean squared error between the model and 16 typical human saccadic eye movements similar to that shown in Fig. 2.

This model is homeomorphic. That is, there is a one for one correspondence between the elements of the model and elements or effects in the physiological system. For example, there is a specific physiological experiment that shows muscle can be lengthened instantaneously by a quick jerk. The series elasticity models the result of this experiment. Homeomorphism also requires that the state equations be written with physical variables. The state variables for the model are

$x_1 = \theta_1$ position of eye,
$x_2 = \theta_2$ position of agonist node, shown in Fig. 1,
$x_3 = \theta_3$ position of antagonist node, shown in Fig. 1,
$x_4 = \dot{\theta}_1$ eye velocity,

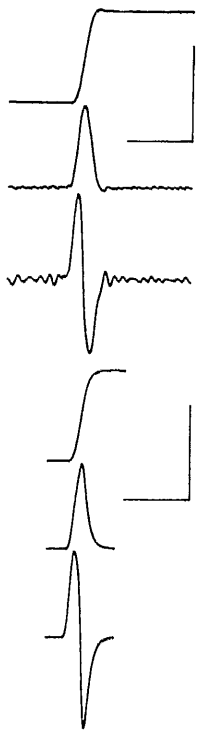


Fig. 2. Human (top) and model (bottom) 10 deg saccadic eye movements showing from top to bottom, as functions of time, eye position, eye velocity, and eye acceleration low-pass filtered, respectively, at 300, 80, and 60 Hz. Calibration bar represents 10 deg, 500 deg/s, and 30,000 deg/s^2. Mean-squared error between human and model saccades is 30×10^{-6} deg^2.

$x_5 = F_{\text{AG}}$ agonist active state tension,
$x_6 = F_{\text{ANT}}$ antagonist active state tension.

The Sensitivity Analysis

The absolute sensitivity function, $S = \delta_y / \delta_\beta$, where y is the output and β is the parameter of interest, is useful for computing parameter induced output errors [3], but is not useful for comparing effects of different parameters. For this comparison we want a relative sensitivity function such as

$$\bar{S} = \frac{\delta \ln y}{\delta \ln \beta} = \frac{\delta y}{\delta \beta} \frac{\beta_0}{y_0}$$

where β_0 and y_0 are the values of β and y at the nominal operating point. However, for a 10 deg saccadic eye movement the nominal output value y_0 varies from 0–10 deg, and division by zero is not possible. Furthermore, the relative sensitivity function gives undue weight to the beginning of the saccade when y_0 is small. Therefore, the semirelative sensitivity function was used:

$$\tilde{S} = \frac{\delta y}{\delta \ln \beta} = \frac{\delta y}{\delta \beta} \beta_0.$$

We have used the semirelative sensitivity function rather than the absolute sensitivity function, because relative and semirelative sensitivity functions do not change their values with changes in units. For example, when the semirelative sensitivity of eye position with respect to pulse height is calculated, its maximum value is 7.1 deg, whereas when the absolute sensitivity of eye position with respect to pulse height is calculated, its maximum value is 0.0431 deg/g if the pulse height is given in units of gram tension and is 4.39 deg/N if the pulse height is given with units of newtons. For small parameter changes in our linear system, the semirelative sensitivity function became

$$\tilde{S} = \frac{\Delta y}{\Delta \beta} \beta_0.$$

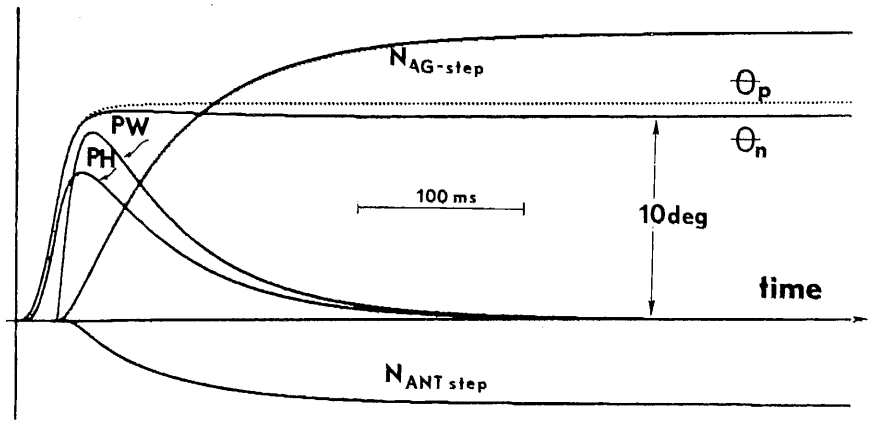


Fig. 3. Nominal (solid line labeled θ_n) and perturbed (dotted line labeled θ_p) 10 deg saccadic eye movements and semirelative sensitivity functions for parameters describing input controller signals. Perturbed saccade shown here is that produced by increasing $N_{\text{AG-step}}$ by 5 percent. Pulsewidth (PW) and pulse height (PH) primarily affect dynamic saccade and behavior immediately following; steady state neural firing levels $N_{\text{AG-step}}$ and $N_{\text{ANT-step}}$ primarily affect static behavior of eye. Effect of $N_{\text{ANT-pulse}}$ is too small to be seen on this scale. Its sensitivity function looks like noise on abscissa. Semirelative sensitivity functions have same units and are plotted to same scale as eye movements. Record length is 490 ms.

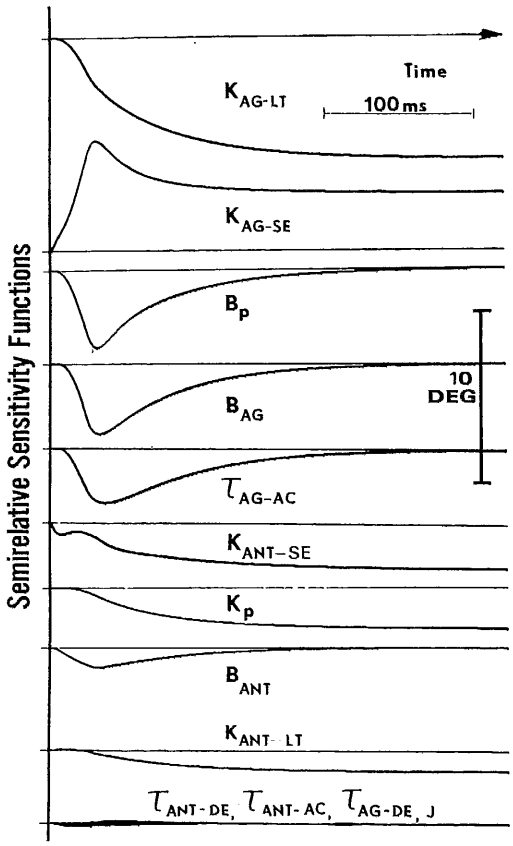


Fig. 4. Semirelative sensitivity functions for other 13 parameters of model. Functions for J, $\tau_{\text{ANT-AC}}$, $\tau_{\text{AG-DE}}$, and $\tau_{\text{ANT-DE}}$ overlap each other and are not distinct on this scale.

To perform the sensitivity analysis, a 10 deg saccade was simulated (solid line labeled θ_n in Fig. 3). Then one parameter was changed by a set amount $\Delta\beta$, +5 percent for these figures, and the model was run again, producing the perturbed saccade (dotted line labeled θ_p in Fig. 3). Each millisecond the difference between the nominal and perturbed saccades (Δy) was calculated and divided by the change in the parameter value ($\Delta\beta$). This ratio was then multiplied by the nominal parameter value (β_0). For the fixed parameter change of +5 percent, the sensitivity function could be written as $\tilde{S}=20\Delta y$. The sensitivity functions calculated for each of the 18 parameters in the model are shown in Figs. 3 and 4.

The model was most sensitive to the input control signals for the agonist muscle, $N_{\text{AG-step}}$, PH, and PW. The control signals for the antagonist, $N_{\text{ANT-step}}$ and $N_{\text{ANT-pulse}}$ were not as important.

In contrast, the model behavior had very little dependence on many other parameters. For instance, the sensitivity of the output to variations of the inertia J was almost zero. This then was the justification for modeling the inertia of the eyeball as a globe of ice rather than as a series of concentric shells connected with viscoelastic elements.

Some of the parameters affected the dynamic properties of the saccade, and some parameters affected primarily the steady state or static behavior, the behavior after the completion of the dynamic saccade. Five of the sensitivity functions of this study were monotonic, such as the sensitivity to $N_{\text{AG-step}}$. Most sensitivity functions were monophasic: the sensitivity for the inertia J was the only biphasic function. The sensitivity of $K_{\text{ANT-SE}}$ was the only function with a relative minimum, as well as different absolute minimum.

Table I compares the maximum values of the 18 sensitivity functions. The first column is a rank ordering of the maximum values of the semirelative sensitivity functions of all 18 parameters of the linear model; the second column is a rank ordering of the sensitivity functions according to their maximum values during and near the end of the saccade (dynamic effects); and the third column is a rank ordering of the sensitivity functions according to their maximum values after the movement was completed and the eye came to rest at the end of the record (static effects).

The following are some of the conclusions that were derived from the sensitivity analysis. When the subscripts AG and ANT are omitted, both elements are indicated.

1) The series elasticities K_{SE} are the only parameters that have important effects on both static and dynamic properties.

2) The parameters N_{step}, K_{LT}, and K_p affect the steady state of $\theta(t)$ and have little affect on either the overshoot or the rise time.

3) The sensitivity functions for the time constants are very small. Although $\tau_{\text{ANT-DE}}$ had its greatest effect during the initial portion of the saccade, it cannot be said to control the rise time.

TABLE I
RANK ORDERING OF SEMIRELATIVE SENSITIVITY FUNCTIONS OF 18 PARAMETERS OF LINEAR HOMEOMORPHIC MODEL

OVERALL	DYNAMIC	STATIC
1 $N_{AG-step}$		1 $N_{AG-step}$
2 PW	1 PW	
3 PH	2 PH	
4 K_{AG-LT}		2 K_{AG-LT}
5 K_{AG-SE}	3 K_{AG-SE}	3 K_{AG-SE}
6 B_p	4 B_p	
7 $N_{ANT-step}$		4 $N_{ANT-step}$
8 B_{AG}	5 B_{AG}	
9 τ_{AG-AC}	6 τ_{AG-AC}	
10 K_{ANT-SE}		5 K_{ANT-SE}
11 K_p		6 K_p
12 B_{ANT}	7 B_{ANT}	
13 K_{ANT-LT}		7 K_{ANT-LT}
14 τ_{ANT-DE}		
15 J		
16 τ_{ANT-AC}		
17 $N_{ANT-pulse}$		
18 τ_{AG-DE}		

The series elasticities have much greater effects on the rise time.

4) The sensitivity functions for the other three time constants and for the three dashpots had shapes that were similar to each other and that peaked near the end of the saccade. This means that trade-offs could be made between these six parameters without affecting the precision of the model.

5) Pulse height (PH) and pulsewidth (PW) primarily affect the output near the end of the saccade and immediately following the saccade. Variations in both of these parameters produce slow drifts after the saccades. These drifts are called *glissades*. However, the shapes of these two sensitivity functions differ. The pulse height function rises gradually, starting at the beginning of the saccade, whereas the pulsewidth sensitivity function is zero until near the end of the saccade, where it abruptly rises to its peak, as shown in Fig. 3. This means that increasing either parameter would produce a larger saccade with a glissade attached to the end. However, increasing the pulse height would also increase the peak velocity of the saccade, whereas increasing the pulsewidth would not affect the peak velocity, because the peak velocity occurs in the middle of the saccade, while this sensitivity function is still zero. Physiological data have shown that saccades with this type of glissade appended have normal or even lower than normal peak velocities [8]. Therefore, the sensitivity analysis explains why glissades are caused by pulsewidth errors and not pulse height errors.

DISCUSSION

Glissades, the slow drifting movements sometimes appended to saccades, occur in fatigued normals and in patients with neurological diseases such as myasthenia gravis and multiple sclerosis [9]. Understanding the generation of glissades may help to explain the effects of these disorders on the human ocular motor systems. Early studies of glissades [7], [10], [11] found no differences between model glissades generated by pulsewidth and pulse height errors. They therefore suggested that glissades could be created by either pulsewidth or pulse height errors. Later model simulations showed that the saccadic peak velocities were different depending upon whether there were pulsewidth or pulse height errors. Subsequent examination of human eye movements showed that "Glissadic overshoots are due to pulsewidth errors" [8]. At this time there was no explanation of why this should be true. Our present sensitivity analysis explains why pulsewidth and not pulse height errors should be responsible for glissadic overshoots, and it also extends the conclusion to include glissadic undershoots. Our sensitivity analysis explains why this is so by showing that eye position is sensitive to pulsewidth variations only near the end of the saccade, after the point of peak velocity. So, pulsewidth changes affect the size of the saccade, but not its peak velocity. On the other hand, eye position is sensitive to pulse height variations throughout the saccade. Therefore, pulse height changes do affect the peak velocity.

Once a model has been constructed, it is natural to ask, "Is it a good model?" Answering this question is assessing the validity of the model. We used five methods to validate our model. First, the position and velocity records of the model qualitatively matched those of humans. Second, the peak velocity magnitude duration (*main sequence*) parameters of the model matched those of 3000 human saccades over a range of 1–40 deg. Third, the average mean squared error between the model and 16 consecutive human saccades was 52×10^{-6} deg^2. This compares with an average mean squared error of 45×10^{-6} deg^2 between the simultaneous saccades of the left and right eyes [6]. Fourth, we used the model to simulate novel eye movements that were not used in developing the model, and then we made specific predictions about the neural signals responsible for these movements. These simulations led to the prediction that glissadic overshoots and glissadic undershoots are due to pulsewidth errors and not to pulse height errors, or a combination of pulse height and pulsewidth errors. Fifth, the sensitivity analysis explained why pulsewidth and not pulse height errors are responsible for glissades. Furthermore, it helped to validate this linear model by showing which parameters were most important and when they had their maximum effect on the output.

Because our sensitivity functions were functions of time, it was easy to see what part of the movement should be studied in order to see the effects of any particular parameter. For example, the time constant τ_{ANT-DE} had its greatest effect on the output early in the saccade, so to study this parameter we studied the beginning of the saccade. If the study had involved visual inspection of the waveforms, then we would have used the model's output to derive some other function, such as the acceleration, that would have highlighted this region. Then in order to study the effects of τ_{ANT-DE} we would have looked at the peak positive acceleration. A similar study of derived parameters has been performed on another eye movement model [5]. This type of analysis adds no new information, because only the eye position is recorded, and these other functions are derived from them. However, it does make it easier for a human to visualize the effects.

For the most part, the rank ordering of our sensitivity functions is similar to the results of Hsu *et al.* [4] and Lehman and Stark [5]. The only exception is the sensitivity to B_{AG}. In our analysis it ranks tenth. In these other studies it ranked second and third. Human physiological data are not available for this parameter. However, it does not play an important role in our model.

This model is neither the simplest nor the most complicated possible model for a sixth-order system. In general, a sixth-order system can be described with a characteristic polynominal containing only seven coefficients, or with a six by six A matrix with

36 coefficients. Our linear model contains 18 parameters. We do not wish to condense the model, because each element of the model corresponds to a real element or an effect in the physiological system: the model is homeomorphic. Homeomorphism makes it possible to use the model to explain which human parameters might have been changed by pathology, fatigue, or drugs to produce certain abnormal human eye movements [9].

Summary

1) Semirelative sensitivity functions should be used when comparing the effects on the step response of the system caused by variations of the different parameters.

2) This sensitivity analysis explains why both glissadic overshoots and undershoots are caused by pulsewidth not pulse height errors.

3) Elements such as the inertia of the globe that have small sensitivity functions can be modeled simply.

4) Elements of this model that are not derived directly from human data, such as the dashpots representing the force–velocity relationships, do not have a great effect on the model.

5) A sensitivity analysis is a powerful tool for studying and validating models.

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