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A technique for validating recursive filters*

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Abstract. A new technique is introduced for verifying computer implementations of recursive digital filters. A target waveform is presented to the input of the filter and the filter coefficients are allowed to converge. Then the values of the coefficients are checked to see if they converged to the correct values. These correct values are derived first for a sinusoidal input. Then a general derivation shows what the coefficients should converge to for a large variety of waveforms.

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1. Introduction

Designing a system means making a model upon which a real world system can be built. Ensuring that the model is a solution to the real world problem is called validation (building the right system.) However, a more important aspect of systems engineering is proving that the computer implementation emulates the model (building the system right). This later verification issue is the topic of this paper.

There are many types of digital filters. The recursive algorithms, such as least–mean–squar (LMS) [1,2] and recursive–least–square (RLS) [3,4], worked well for our application. While investigating these filters we developed a technique for proving that the computer implementation was correct. This technique can be generalized to all types of recursive filters.

The conventional way of verifying adaptive filtering algorithms can be described as follows.

1. Use simulation to approximate coefficients of a signal model.

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2. Fed the signal generated by the model to the adaptive filter and allow the coefficients of the filter to converge.
3. Compare the coefficients of the filter with those of the model.

In this paper, we describe a new technique for verifying recursive least squares adaptive filters. The procedures of this technique can be summarized as follows.

1. Choose a signal, a sinusoid for example. This signal can be described by a difference equation in terms of its past values and a set of autoregressive coefficients.
2. Synthesize a model whose output is the chosen signal.
3. Pre–compute the exact values of the coefficients of the chosen signal using the technique introduced in this paper.
4. Feed the signal generated by the model to the adaptive filter and allow the coefficients of the filter to converge.
5. Compare coefficients of the filter with those of the model.

Our technique is better than the conventional algorithm in three ways. First, we have a mathematical way of deriving coefficients of the chosen signal in terms of the n–th order difference equation. Second, we use a simple signal, such as an impulse or a step, as the input to the adaptive filter. This allows us to easily derive coefficients of the chosen signal. This is unlike the conventional technique where a complicated signal is the input to the adaptive filter. With the conventional technique it is difficult to mathematically verify the coefficients of the chosen signal. Third, the coefficients of a signal can be derived theoretically and proved by computer simulation. Unlike in the conventional technique, where the coefficients of the signal are chosen arbitrarily without theoretical support.

Modeling, identification, filtering predicting, and tracking are closely related activities. If you identify a signal, you have modeled it. If you can model it, you can predict its future output. If you can predict its future output, you can track it. Therefore, any digital filter can also be used for prediction. So, although our specific problem was that of prediction, in this paper we make little distinction between these terms.

2. Least square estimators

Let us first present a general review of the well known RLS technique. The recursive–least–square predictor (RLSP) is a parameter identification scheme that can be described by a difference equation of the form

$$y(t_i) = a_1 y(t_{i-1}) + a_2 y(t_{i-2}) + \ldots + a_N y(t_{i-N}). \tag{1}$$

Equation (1) is called an autoregressive model. In this model, the current value of the process $y(t_i)$ is expressed as a finite, linear aggregate of previous values of the process and a set of autoregressive coefficients, the a_k's, where $k = 1, \ldots, N$ and N is the order of the signal (the number of coefficients needed to describe the signal). The RLSP generates $\hat{a}_1$ to $\hat{a}_N$ that are the estimates of the autoregressive coefficients, a_1 through a_N. Thus, it yields a model of the signal given by

$$\hat{y}(t_i) = \hat{a}_1 y(t_{i-1}) + \hat{a}_2 y(t_{i-2}) + \ldots + \hat{a}_N y(t_{i-N}). \tag{2}$$

Here, $\hat{y}(t_i)$ is an estimated value of $y(t_i)$ based on the parameters $\hat{a}_k$, $(k = 1, \ldots, N)$. The RLSP minimized the sum of squared errors:

$$J = \sum_{i=0}^{N} [y(t_i) - \hat{y}(t_i)]^2 . \tag{3}$$

The squared error is chosen as the cost function, J, in the minimization procedure because it is differentiable and non-negative. Minimizing squared error leads to the name "least squares." The recursive formulae can be described as

$$\underline{\hat{\Theta}}(t_N) = \underline{\hat{\Theta}}(t_{N-1}) + \underline{L}(N) \left[y(t_N) - \underline{\Phi}^T(t_N) \underline{\hat{\Theta}}(t_{N-1}) \right] , \tag{4}$$

where $\underline{\hat{\Theta}}(t_N)$ is a vector of parameters,

$$\underline{\hat{\Theta}}(t_N) = [\hat{a}_1(t_N), \hat{a}_2(t_N), \ldots, \hat{a}_N(t_N)]^T , \tag{5}$$

$\underline{\Phi}(t_i)$ is a vector of past signal values,

$$\underline{\Phi}(t_i) = [y(t_{i-1}), y(t_{i-2}), \ldots, y(t_{i-N})]^T , \tag{6}$$

and

$$\underline{L}(N) = \frac{\underline{P}(N-1)\underline{\Phi}(t_N)}{I + \underline{\Phi}^T(t_N)\underline{P}(N_1)\underline{\Theta}(t_N)} . \tag{7}$$

The matrix $\underline{P}(N)$ is updated with

$$\underline{P}(N) = \underline{P}(N-1) - \underline{L}(N)\underline{\Phi}^T(t_N)\underline{P}(N-1). \tag{8}$$

To start the process simply let $\underline{P}(0) = \sigma \underline{I}$, where σ is a large (positive) number and let $\underline{\hat{\Theta}}(0) = 0$. When the implementation converges, estimators $\hat{y}(t_i)$ are almost identical to the correct values $y(t_i)$ for large N.

3. Accuracy of the LRS algorithm

When an RLS algorithm is adapting to a new waveform its coefficients will be changing. When both the waveform and the algorithm have reached steady–state the coefficients will no longer change, up to a small error tolerance. The exact coefficients are unique for each input waveform. Finding these coefficients is equivalent to identifying the signal. For some waveforms these coefficients can be calculated theoretically. Therefore, the coefficients derived by the algorithm can be compared to these theoretical coefficients to evaluate the accuracy of the computer implementation of the algorithm.

To understand our verification technique, assume that a signal is coming out of a black box. The behavior of the black box can be described by a difference equation with the transfer function H. Now the identification task is to estimate the transfer function $\widehat{H}$ of the black box. For our specific case, the desired output from the RLSP is the predicted signal, $Y(z^{-1})$. To simplify the mathematical derivations, choose the input signal, $U(z^{-1}) = 1$, to be an impulse and the system transfer function, $H(z^{-1})$, becomes equal to $Y(z^{-1})$. This combination of the input signal and system produces an output signal,

$$Y(z^{-1}) = U(z^{-1})H(z^{-1}) \tag{9}$$

as desired.

4. Specific formulas

4.1. Sinusoidal signal

Now assume that the signal, $y(t)$, coming out of the black box is sinusoid. The identification task is to find the transfer function H that would produce such an output in response to a unit impulse input. Consider a sinusoidal position signal of frequency ω, the position can be described as $\sin \omega t$, which is the output signal $Y(z^{-1})$ in the time domain, so in the z–domain,

$$H(z^{-1}) = \frac{Y(z^{-1})}{U(z^{-1})} = Y(z^{-1}). \tag{10}$$

That is,

$$(11) \qquad H(z^{-1}) = \sum_{t=0}^{\infty} \sin \omega t \, z^{-t} = \sum_{t=0}^{\infty} \frac{e^{j\omega t} - e^{-j\omega t}}{2j} z^{-t}$$
$$= \frac{1}{2j}\left[\left(\sum_{t=0}^{\infty} e^{j\omega t} \, z^{-t}\right) - \left(\sum_{t=0}^{\infty} e^{-j\omega t} \, z^{-t}\right)\right].$$

Hence

$$(12) \qquad H(z^{-1}) = \frac{z \sin \omega}{z^2 - 2z \cos \omega + 1}$$
$$(13) \qquad = \frac{z^{-1} \sin \omega}{1 - 2z^{-1} \cos \omega + z^{-2}}.$$

Combining equations (10) and (13) yields

$$(14) \qquad Y(z^{-1})(1 - 2z^{-1} \cos \omega + z^{-2}) = U(z^{-1})(z^{-1} \sin \omega).$$

Taking the inverse of the z–transformation of equation (14) yields, in the time domain,

$$(15) \qquad y(t_i) - 2 \cos \omega y(t_{i-1}) + y(t_{i-2}) = \sin \omega u(t_{i-1}).$$

Since the input is an impulse, $u(t_i) = 0$, for $i > 0$. Thus, for $i > 1$,

$$(16) \qquad y(t_i) = 2 \cos \omega y(t_{i-1}) + y(t_{i-2}) = 0.$$

That is,

$$(17) \qquad y(t_i) = 2 \cos \omega y(t_{i-1}) - y(t_{i-2}).$$

Comparing equation (17) with the standard form (2nd–order difference equation)

$$(18) \qquad y(t_i) = a_1 y(t_{i-1}) + a_2 y(t_{i-2}),$$

yields

$$(19) \qquad a_1 = 2 \cos \omega, \quad \text{and} \quad a_2 = -1.$$

Equation (19) shows that a sinusoidal signal can be described as a second–order difference equation and its autoregressive coefficients a_1 and a_2 will converge to $2 \cos \omega$ and -1 respectively. Therefore, if a black box containing a second order difference equation has a sinusoidal output with frequency ω, then the difference equation must be of the form

$$(20) \qquad y(t_i) = 2 \cos \omega y(t_{i-1}) - y(t_{i-2}).$$

To illustrate that this equation really produces a sinusoid, let the period, P, equal 10, meaning $\omega = 0,63$ and $2\cos\omega = 1.62$, and start off with initial conditions of 0 and 1. Equation (20) will produce the sinusoid of Table 1, with a period of 10.

Table 1. A sinusoid produced by Eq (20)

t	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$y(t)$	0	1	1.62	1.62	1	0	-1	-1.62	-1.62	-1	0	1	1.62	1.62	1	0	1

So to test our computer implementation of the RLS algorithm, we gave it a sinusoidal signal and checked to see if its coefficients converged to $2\cos\omega$ and -1. In one computer simulation, the period of the signal, P, was chosen to be 10. ω equals $2\pi/P$, so, $2\cos\omega$ was 1.61803. The simulations results of the RLSP showed the coefficients converged to 1.61998 and -0.99998, which is pretty close to the ideal values of 1.61803 and -1.

4.2. Other test waveforms

We also used cubic, parabolic and triangular waveforms to test our algorithms. For example, the cubic position signal was described as

$$y(t) = 10.39A\left[2\left(\frac{t}{P}\right)^3 - 3\left(\frac{t}{P}\right)^2 + \frac{t}{P}\right], \quad 0 < t < P, \tag{21}$$

where A is amplitude and P is period. We repeated this curve every P seconds to make it a periodic signal. The parabolic position signal was described by

$$y(t) = \begin{cases} A\left[+1 - \left(\frac{t-\frac{P}{4}}{\frac{P}{4}}\right)^2\right], & \text{if } 0 \le t \le \frac{P}{2} \\ A\left[-1 + \left(\frac{t-\frac{P}{4}}{\frac{P}{4}}\right)^2\right], & \text{if } \frac{P}{2} < t \le P \end{cases}. \tag{22}$$

The triangular position signal was expressed as

$$y(t) = \begin{cases} +\frac{2A}{P}t, & \text{if } 0 \le t \le \frac{P}{2} \\ -\frac{2A}{P}t + 2A, & \text{if } \frac{P}{2} < t \le P \end{cases}. \tag{23}$$

This collection of equations provided a set of waveforms that were similar in shape to the sinusoids, yet they had distinctive velocity waveforms and spectral characteristics [5]. We derived the autoregressive coefficients for these waveforms in a manner similar to the sinusoids in the previous section. The results are shown in Table 2.

Table 2
Theoretical coefficients for the difference equation

Waveform	a_1	a_2	a_3	a_4	order
Sinusoid	$2\cos\omega$	-1	0	0	2
Cubic	4	-6	4	-1	4
Parabolic	3	-3	1		3
Triangular	2	-1			2
Square–wave	1				1

However, rather than show individual derivations for these special cases, we shall now show a general derivation for any waveform.

5. General derivation

For a general difference equation of the form

$$y(t_i) = a_1 y(t_{i-1}) + a_2 y(t_{i-2}) + \ldots + a_N y(t_{i-N}), \tag{24}$$

the characteristic polynomial is defined as

$$\lambda^N - a_1\lambda^{N-1} - a_2\lambda^{N-2} - \ldots - a_{N-1}\lambda - a_N = 0. \tag{25}$$

Let the roots be $\lambda_1, \lambda_2, \ldots, \lambda_r$ with multiplicities $m_1, m_2, \ldots, m_r$, respectively. Here necessarily $m_1 + m_2 + \ldots + m_r = N$. Then the general solution has the form:

$$y(t_i) = \sum_{\ell=1}^{r} \sum_{s=0}^{m_\ell - 1} c_{\ell s} i^s \lambda_\ell^i. \tag{26}$$

That is,

$$y(t_i) = \sum_{\ell=1}^{r} y_\ell(i), \tag{27}$$

where

$$y_\ell(i) = \left[c_{\ell 0} + c_{\ell 1} i + c_{\ell 2} i^2 + \ldots + c_{\ell, m_\ell - 1} i^{m_\ell - 1}\right] \lambda_\ell^i. \tag{28}$$

Assume first that for some ℓ, $\lambda_\ell = 1$, then

$$y_\ell(i) = c_{\ell 0} + c_{\ell 1} i + c_{\ell 2} i^2 + \ldots + c_{\ell, m_\ell - 1} i^{m_\ell - 1}, \tag{29}$$

since $\lambda_\ell^i \equiv 1$ for all i. Hence, if $m_\ell = 2$, the triangular part is obtained, if $m_\ell = 3$, the parabolic part is obtained, and if $m_\ell = 4$, the cubic part is obtained.

Assume next that for some $\ell, \lambda_\ell = (\cos\omega + j \sin\omega)\rho$ (complex roots). Then

$$y_\ell(i) = \rho^i(\cos i\omega + j \sin i\omega)(c_{\ell 0} + c_{\ell 1} i + c_{\ell 2} i^2 + \ldots + c_{\ell, m_\ell - 1} i^{m_\ell - 1}), \tag{30}$$

which gives the trigonometric polynomial form. In the future special case, when $m_\ell = 1$ (multiplicity 1),

$$y_\ell(i) = \rho^i(\cos i\omega + j \sin i\omega) c_{\ell 0}. \tag{31}$$

The polynomial part vanishes, hence we get the trigonometric form. The results of this general derivation are summarized in Tables 2 and 3.

Table 3
Roots and multiplicities of general scheme

Waveform	Roots (λ_ℓ)	Multiplicities (m_ℓ)
Sinusoid	$(\cos\omega + j \sin\omega)\rho$	1
Cubic	1	4
Parabolic	1	3
Triangular	1	2
Square	1	1

So, we have shown that some trigonometric and polynomial waveforms can be obtained from higher order difference equations, as given in Table 2, by carefully selecting initial values so that the unwanted parts of the general solution cancel out.

6. General results

When one of these waveforms is applied to the input of a recursive filter its coefficients should (if it works right) converge to the values shown in Table

2. A sinusoidal signal can be described as a second–order difference equation and its autoregressive coefficients a_1 and A_2 will converge to $2\cos\omega$ and -1 respectively. A cubic signal can be described as a fourth–order difference equation and its autoregressive coefficients a_1, a_2, a_3, and a_4 will converge to $4, -6, 4$, and -1 respectively. A parabolic signal can be described as a third–order difference equation with autoregressive coefficient a_1, a_2, and a_3 to be $3, -3$, and 1 respectively. Finally, a triangular signal can be also described as a second–order difference equation with autoregressive coefficient a_1 and a_2 to be 2 and -1 respectively.

Our computer implementation of the RLSP worked well as was demonstrated by its coefficients, shown in Table 4, converging to the correct values of Table 2.

Table 4
Actual coefficients for our implementation

Waveform	a_1	a_2	a_3	a_4
Sinusoid	1.61998	–0.99998	0	0
Cubic	3.99998	–5.99998	3.99998	–0.99998
Parabolic	2.99998	–2.99998	0.99998	
Triangular	1.99998	–0.99998		
Square–wave	0.99998			

Four conditions are necessary for accurate predictions using an RLS predictor. (1) The order of the model must be greater than or equal to the order of the input signal: If it is not, the algorithm will not converge. When we run the RLSP on the computer we assign a value to the order of the RLSP. If the order of the input signal is unknown we assign a large number. If the order of the input signal is known (as for the signals in Table 2) we assign the exact value. (2) The parameter estimates, $\widehat{a}_1$ through $\widehat{a}_N$, must converge to the actual parameters of the input signal, as shown in Table 4. (3) The signal parameters must be time invariant. For example, for a sinusoid $a_1 = 2\cos\omega$, and $a_2 = -1$. As you can see a_1 and a_2 are time invariant. If the signal parameters vary with time the RLSP will not converge. (4) The z–transform of the input signal must have at least one pole. The denominator of equation (12), $z^2 - 2z\cos\omega + 1$, is the characteristic function. If we set this equal to zero we get the characteristic equation. For a sinusoid this characteristic equation has two poles. If no poles exist in the z–transform of the input signal, then characteristic equation is constant. Therefore there is no difference equation in the time domain.

7. Summary

In this paper we presented a technique for verifying computer implementations of recursive digital filters. Target waveforms were presented to the input of the filter and the filter coefficients were allowed to converge. If they converged to the correct values we concluded that the computer implementation was correct. These correct values were derived for sinusoidal, cubic, parabolic and triangular waveforms. Our technique is general sot that any desired waveform could be used to verify a computer implementation of a recursive filter.

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