PU.M.A. Ser. B, Vol. 1 (1990), No. 2-3, pp. 115-121

On stable adaptive control systems [1]

F. SZIDAROVSZKY[2], T.A. BAHILL[2] AND S. MOLNÁR[3]

Abstract. A necessary and sufficient condition is presented for the existence of asymptotically stable adaptive control systems. This condition is based on the solvability of certain system of nonlinear algebraic equations. A numerical example illustrates the theoretical result.

AMS Subject Classifications. 93C40, 93D20.

1. Introduction

In recent years increasing attention has been given to systems that are capable of accommodating unpredictable changes, whether these changes arise within the system or externally. This property is called *adaptation* and is a fundamental characteristic of living organisms, since they attempt to maintain physiological equilibrium in order to survive under changing environmental conditions. In the system theory literature there is no unified definition for adaptive control systems. Therefore, as in Landau (1979), we will consider a system adaptive if it satisfies the following criteria:

1. Continuously and automatically measures the dynamic characteristics of the system;

2. Compares the measurements to the desired dynamic characteristics;

3. Modifies it own parameters in order to maintain desired performance regardless of the environmental changes.

An adaptive control system therefore consists of three blocks: performance index measurement, comparison–decision, and adaptation mechanism. It is always assumed that there is a closed–loop control on the performance index. An important class of adaptive systems, *model reference*

[1]Received: November 10, 1990.

[2]Systems and Industrial Engineering Department, University of Arizona, Tucson, Arizona, 85721, U.S.A.

[3]Central Mining Development Institute, Budapest, Mikoviny S. utca 2-4, H-1037, Hungary

adaptive systems, is based on replacing the set of given performance indices by a reference model.

The output of this model and that of the adjustable system are continuously compared by a typical feedback comparator, and the difference is used by the adaptation mechanism either to modify the parameters of the adjustable system or to send an auxiliary input signal to minimize the difference between the performance indices of the two systems.

Another often used class of adaptive systems is given by the *adaptive model-following control systems*. These control systems also use a model that specifies the design objectives as it is illustrated in the Figure.

In this paper this second type of adaptive systems is examined. A mathematical model is first presented, and the stability of the resulting adaptive system is investigated. Namely, we introduce a new necessary and sufficient condition for the existence of globally asymptotically stable adaptive model–following control systems.

The mathematical model is formulated as follows. Assume that the reference model is given as

$$\dot{\mathbf{x}} = \mathbf{A}_M\mathbf{x} + \mathbf{B}_M\mathbf{u}_M\,, \tag{1}$$

and the plant to be controlled is

$$\dot{\mathbf{y}} = \mathbf{A}_P\mathbf{y} + \mathbf{B}_P\mathbf{u}_P\,. \tag{2}$$

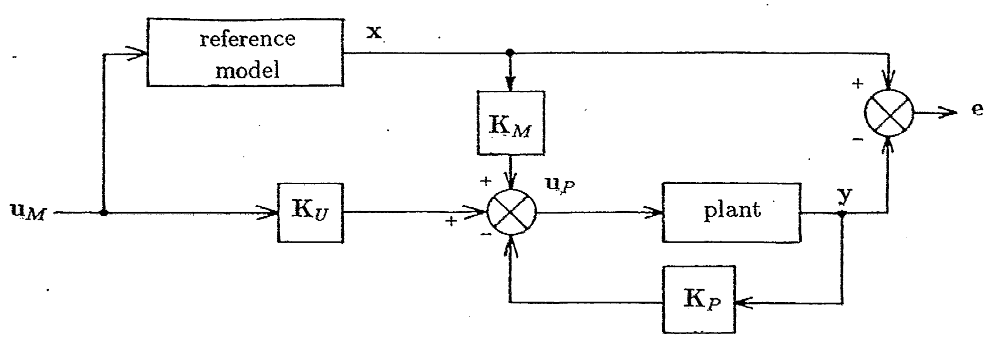

Figure 1: Linear adaptive model-following control system

The plant control input is given by the relation

$$\mathbf{u}_P = -\mathbf{K}_P\mathbf{y} + \mathbf{K}_M\mathbf{x} + \mathbf{K}_U\mathbf{u}_M\,. \tag{3}$$

In this formulation $\mathbf{A}_M$, $\mathbf{B}_M$, $\mathbf{A}_P$, $\mathbf{B}_P$ are given constant matrices, x and y are the states of the reference model and the plant, and $\mathbf{u}_M$ and $\mathbf{u}_P$ are their inputs. The coefficient matrices $\mathbf{K}_P$, $\mathbf{K}_M$ and $\mathbf{K}_U$ are unknowns; they are defined so that if the error vector $\mathbf{e} = \mathbf{x} - \mathbf{y}$ is initialized as $\mathbf{e}(0) = \mathbf{0}$, then it remains zero for all future time periods. We can subtract equation (2) from (1) and substitute relation (3) to obtain the following inhomogeneous differential equation:

$$\begin{aligned} \mathbf{e} &= (\mathbf{A}_M - \mathbf{B}_P\mathbf{K}_M)\,\mathbf{e} + (\mathbf{A}_M - \mathbf{A}_P + \mathbf{B}_P(\mathbf{K}_P - \mathbf{K}_M))\,\mathbf{y} \\ &+ (\mathbf{B}_M - \mathbf{B}_P\mathbf{K}_U)\,\mathbf{u}_M\,. \end{aligned} \tag{4}$$

Perfect model following requires, therefore, that

$$\begin{aligned} \mathbf{A}_M - \mathbf{A}_P + \mathbf{B}_P(\mathbf{K}_P - \mathbf{K}_M) &= \mathbf{0} \\ \mathbf{B}_M - \mathbf{B}_P\mathbf{K}_U &= \mathbf{0}\,, \end{aligned} \tag{5}$$

since these equations imply that for all real vectors $\mathbf{y}$ and $\mathbf{u}_M$ of appropriate dimensions, equation (4) becomes homogeneous, and so, the solution of the resulting homogeneous equation with zero initial condition is the zero vector for all $t \geq 0$. We can rewrite equations (5) as

$$\begin{aligned} \mathbf{B}_P(\mathbf{K}_P - \mathbf{K}_M) &= \mathbf{A}_P - \mathbf{A}_M \\ \mathbf{B}_P\mathbf{K}_U &= \mathbf{B}_M\,. \end{aligned} \tag{6}$$

The necessary and sufficient condition for the existence of matrices $\mathbf{K}_P$, $\mathbf{K}_M$ and $\mathbf{K}_U$ that satisfy equations (6) is the following:

$$\operatorname{rank}(\mathbf{B}_P) = \operatorname{rank}(\mathbf{B}_P,\, \mathbf{A}_P - \mathbf{A}_M) = \operatorname{rank}(\mathbf{B}_P,\, \mathbf{B}_M)\;. \tag{7}$$

These conditions mean that all columns of both matrices $\mathbf{A}_P - \mathbf{A}_M$ and $\mathbf{B}_M$ are in the subspace spanned by the columns of matrix $\mathbf{B}_P$. Note that equations (6) can be solved using Gauss–elimination (see, for example, Szidarovszky and Yakowitz, 1978).

Usually, the initial condition of the error vector $\mathbf{e}$ differs from zero. In such cases we require that $\mathbf{e}(t) \longrightarrow 0$ as $t \longrightarrow \infty$, that is, equation (4) is asymptotically stable. We know (see, for example, Kailath, 1980) that this additional condition holds if and only if all eigenvalues of matrix $\mathbf{A}_M - \mathbf{B}_P\mathbf{K}_M$ have negative real parts.

2. Stability analysis

It is well known from the system theory literature (see, for example, Szidarovszky and Bahill, 1991) that there exists a matrix $\mathbf{K}_M$ such that all eigenvalues of $\mathbf{A}_M - \mathbf{B}_P\mathbf{K}_M$ have negative real parts if the modified controllability matrix $\left(\mathbf{B}_P, \mathbf{A}_M\mathbf{B}_P, \ldots, \mathbf{A}_M^{n-1}\mathbf{B}_P\right)$ has full rank. This condition is sufficient, but not necessary as the following example illustrates.

Example 1.. Define $n = 2$,

$$\mathbf{A}_P = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \mathbf{B}_P = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix},$$

$$\mathbf{A}_M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{B}_M = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}.$$

Here the modified controllability matrix

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

has unit rank, so the above condition does not hold. However, we can easily find a $\mathbf{K}_M$ such that equations (6) hold and all eigenvalues of $\mathbf{A}_M - \mathbf{B}_P\mathbf{K}_M$ have negative real parts.

Note first, that equations (6) can be rewritten as

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix},$$

where matrix $\mathbf{K}_P - \mathbf{K}_M$ is denoted by (r_{ij}) and $\mathbf{K}_U$ is denoted as (k_{ij}). Expanding the above operations, we get the following system of linear equations:

$$\begin{aligned} r_{11} + 2r_{21} &= 1 \\ r_{12} + 2r_{22} &= 1 \\ k_{11} + 2k_{21} &= 2 \\ k_{12} + 2k_{22} &= 1, \end{aligned}$$

where the repeated equations are omitted. It is easy to see that $r_{11} = r_{12} = 1$, $k_{11} = 2$, $k_{12} = 1$, $r_{21} = r_{22} = k_{21} = k_{22} = 0$ solve these equations. Hence we may select

$$\mathbf{K}_P - \mathbf{K}_M = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_U = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}.$$

There are still infinitely many possibilities for selecting matrix $\mathbf{K}_M$, since only $\mathbf{K}_P - \mathbf{K}_M$ is specified. We wish to make this selection so that matrix $\mathbf{A}_M - \mathbf{B}_P\mathbf{K}_M$ has eigenvalues with only negative real parts. For example, select $\mathbf{K}_M$ so that

$$\mathbf{A}_M - \mathbf{B}_P\mathbf{K}_M = -\mathbf{I},$$

that is,

$$\mathbf{B}_P\mathbf{K}_M = \mathbf{A}_M + \mathbf{I}.$$

If $\mathbf{K}_M = \left(\overline{k}_{ij}\right)$, then this equation has the form

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \overline{k}_{11} & \overline{k}_{12} \\ \overline{k}_{21} & \overline{k}_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

It is easy to see that $\overline{k}_{11} = 1$, $\overline{k}_{12} = 1$, $\overline{k}_{21} = \overline{k}_{22} = 0$ are solutions.

Therefore the selection of

$$\mathbf{K}_U = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{K}_M = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{K}_P = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$$

is satisfactory in order to construct an asymptotically stable adaptive system.

Next, a necessary and sufficient condition is derived for the existence of a suitable matrix $\mathbf{K}_M$. Assume that the rank condition (7) is satisfied. Then equations (6) have solutions for $\mathbf{K}_P - \mathbf{K}_M$ and $\mathbf{K}_U$. Since $\mathbf{K}_P$ can be arbitrary, no constraint is needed for $\mathbf{K}_M$. We know that all eigenvalues of $\mathbf{A}_M - \mathbf{B}_P\mathbf{K}_M$ have negative real parts if and only if equation

$$(\mathbf{A}_M - \mathbf{B}_P\mathbf{K}_M)^T \mathbf{Q} + \mathbf{Q}\,(\mathbf{A}_M - \mathbf{B}_P\mathbf{K}_M) = -\mathbf{I} \tag{8}$$

has positive definite solution $\mathbf{Q}$ where $\mathbf{I}$ is the identity matrix (see, for example, Szidarovszky and Bahill, 1991). Equation (8) is a necessary and sufficient condition, where no restriction is given for $\mathbf{K}_M$, but $\mathbf{Q}$ has to be positive definite. Matrix $\mathbf{Q}$ is positive definite if and only if it can be decomposed as $\mathbf{Q} = \mathbf{L}\,\mathbf{D}\,\mathbf{L}^T$, where $\mathbf{D}$ is a diagonal matrix with positive diagonal, and $\mathbf{L}$ is lower triangular with unit diagonal elements. Next we show that it is sufficient to assume that the diagonal elements of $\mathbf{D}$ are only nonnegative. This observation follows immediately from the fact that any solution $\mathbf{Q}$ of equation (8) is necessarily nonsingular. In contrary to the assertion assume that $\mathbf{Q}$ is singular. Then there exists a real nonzero vector $\mathbf{v}$ such that $\mathbf{Q}\,\mathbf{v} = \mathbf{0}$. Then equation (8) implies that

$$0 > -\mathbf{v}^T\mathbf{v} = \mathbf{v}^T\,(\mathbf{A}_M - \mathbf{B}_P\mathbf{K}_M)^T\,\mathbf{Q}\mathbf{v} + \mathbf{v}^T\mathbf{Q}\,(\mathbf{A}_M - \mathbf{B}_P\mathbf{K}_M)\,\mathbf{v} = \mathbf{0},$$

which is impossible. The nonnegativity of the diagonal elements of $\mathbf{D}$ can be guaranteed by assuming that they are squares of real numbers. Hence we proved the following

THEOREM 1. *There exists an asymptotically stable adaptive model-following control system* (1), (2), (3) *if and only if the rank condition* (7) *holds and there exist a matrix* $\mathbf{K}_M$, *a diagonal matrix* $\mathbf{D}$, *and a lower triangular matrix* $\mathbf{L}$ *with zero diagonal such that*

$$(9)\qquad \begin{aligned}&(\mathbf{A}_M - \mathbf{B}_P\mathbf{K}_M)^T(\mathbf{L}+\mathbf{I})\mathbf{D}^2(\mathbf{L}+\mathbf{I})^T + \\ &+(\mathbf{L}+\mathbf{I})\mathbf{D}^2(\mathbf{L}+\mathbf{I})^T(\mathbf{A}_M - \mathbf{B}_P\mathbf{K}_M) = -\mathbf{I}.\end{aligned}$$

Example 2. In the case of the previous example the rank condition obviously holds, and equation (9) has the form

$$\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - \begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}\begin{pmatrix}\overline{k}_{11} & \overline{k}_{12}\\ \overline{k}_{21} & \overline{k}_{22}\end{pmatrix}\right]^T \begin{pmatrix}1 & 0\\ \ell_{21} & 1\end{pmatrix}\begin{pmatrix}d_{11}^2 & 0\\ 0 & d_{22}^2\end{pmatrix}\begin{pmatrix}1 & \ell_{21}\\ 0 & 1\end{pmatrix} +$$

$$+\begin{pmatrix}1 & 0\\ \ell_{21} & 1\end{pmatrix}\begin{pmatrix}d_{11}^2 & 0\\ 0 & d_{22}^2\end{pmatrix}\begin{pmatrix}1 & \ell_{21}\\ 0 & 1\end{pmatrix}\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - \begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}\begin{pmatrix}\overline{k}_{11} & \overline{k}_{12}\\ \overline{k}_{21} & \overline{k}_{22}\end{pmatrix}\right] =$$

$$= \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}.$$

Simple calculation shows that this matrix equation is equivalent to the following system of algebraic equations:

$$d_{11}^2\left[\ell_{21} - \left(\overline{k}_{11} + 2\overline{k}_{21}\right)(\ell_{21}+1)\right] = -\frac{1}{2}$$

$$d_{11}^2 + \ell_{21}^2 d_{11}^2 + d_{22}^2 - d_{11}^2\left(\overline{k}_{12} + 2\overline{k}_{22}\right)(\ell_{21}+1) - \left(\overline{k}_{11} + 2\overline{k}_{21}\right)$$
$$\left(\ell_{21}d_{11}^2 + \ell_{21}^2 d_{11}^2 + d_{22}^2\right) = 0$$

$$d_{11}^2\ell_{21} - \left(\overline{k}_{12} + 2\overline{k}_{22}\right)\left(\ell_{21}d_{11}^2 + \ell_{21}^2 d_{11}^2 + d_{22}^2\right) = -\frac{1}{2}.$$

We have three equations for seven unknowns, therefore infinitely many solutions exist. Simple substitution shows that

$$\overline{k}_{11} = \overline{k}_{12} = 1\,, \qquad \overline{k}_{21} = \overline{k}_{22} = 0\,,$$

$$\ell_{21} = 0\,, \qquad d_{11}^2 = d_{22}^2 = \frac{1}{2}$$

is a solution. That is,

$$\mathbf{K}_M = \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}$$

is satisfactory. Matrices $\mathbf{K}_P$ and $\mathbf{K}_U$ can be obtained from equations (6), as in the previous example.

3. Conclusion

The results of this paper can be applied in designing practical adaptive control systems. The user must solve equation (9) in order to prove the existence of globally asymptotically stable design, and to find the suitable coefficient matrices.

References

[1] KAILATH, TH. (1980), *Linear systems*, Prentice Hall, Englewood Cliffs, N.J.

[2] LANDAU, Y.D. (1979), *Adaptive control*, Marcel Decker, New York.

[3] MOLNÁR, S. (1990), *A special decomposition of linear systems*, Belgian Journal of Operation Research, Statistics and Computer Science, Vol. 29, No 4, pp 5–37.

[4] SZIDAROVSZKY, F., T.A. BAHILL (1991), *Dynamic systems*, MacMillan, New York, (in press).

[5] SZIDAROVSZKY, F., S. YAKOWITZ (1978), *Principles and procedures of numerical analysis*, Plenum, New York, London.