

## Frequency Limitations of the Two-Point Central Difference Differentiation Algorithm

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**Abstract.** A two-point central difference algorithm is often used to calculate the derivative of a function. This estimate is only valid over a limited frequency range. Therefore, the algorithm can be modeled as an ideal differentiator in series with a low-pass filter. The filter cutoff frequency is a function of the time between the points. We discuss the accuracy and limitations of using this algorithm on human saccadic eye movement data. To calculate the velocity of saccadic eye movements the algorithm should have a cutoff frequency of 74 Hz or above.

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### Frequency Characteristics of the Two-Point Central Difference Algorithm

Computing derivatives is an important facet of many data analysis tasks. It is usually necessary to low-pass filter these derivatives to remove high-frequency noise. An ideal differentiator would require a separate low-pass filter. However, the two-point central difference algorithm differentiates *and* filters. This algorithm is popular because of its speed, simplicity, and accuracy. More complicated derivative algorithms exist (Young and Gregory, 1972; Rabiner and Gold, 1975; Hamming, 1977; Marble et al., 1981; Tole et al., 1978), however, they are not necessarily better. One experimental comparison of five derivative algorithms found the two-point central difference algorithm to be the most accurate technique for 12 bit data (Marble et al., 1981). Usui and Amidror (1981) investigated several integer arithmetic derivative algorithms and found the best to be a two-point central difference algorithm with a built in digital low-pass filter. In this paper we examine the frequency response of the two-point central difference algorithm.

For sinusoids, the magnitude of the derivative increases linearly with frequency. For example, if  $x(t) = \sin \omega t$ , then its time derivative  $\dot{x}(t) = \omega \cos \omega t$ . It is

often convenient to study the Fourier transform of  $\dot{x}(t)$  which is  $j\omega$  times the Fourier transform of  $x(t)$

$$\dot{X}(\omega) = j\omega X(\omega). \quad (1)$$

A two-point central difference algorithm produces similar results only up to a certain frequency. To investigate this algorithm's frequency characteristics, let us represent it in the following form

$$\dot{y}(kT) = \frac{y([k+1]T) - y([k-1]T)}{2T}. \quad (2)$$

$T$  is the sampling interval and  $k$  is the index for discretized time, i.e.,  $k=0, 1, 2, 3, \dots, n$ . The Z-transform of (2) is

$$\dot{Y}(z) = \frac{Y(z)(z - z^{-1})}{2T},$$

where  $\dot{Y}(z)$  represents the Z-transform of  $\dot{y}(kT)$ . We can examine the frequency response by substituting  $z = e^{j\omega T} = \cos \omega T + j \sin \omega T$ . This produces

$$\dot{Y}(\omega T) = \frac{Y(\omega T)j \sin \omega T}{T}. \quad (3)$$

Equation (3) is the frequency response of the derivative calculated with a two-point central difference algorithm; it is plotted in Fig. 1a. Figure 1a and 1b show the comparison of the ideal derivative and the derivative calculated with the two-point central difference algorithm. For this figure and for the rest of this paper we assume a one kHz sampling rate. For a sampling rate of  $N$  kHz, our numbers can be multiplied by  $N$ .

Equations (1) and (3) can be used to compute the ratio of the magnitudes of the calculated derivative to the true derivative. If the data have been correctly low-pass filtered with analog filters to prevent aliasing, then the Fourier transform of the continuous signal,  $X(\omega)$ , is equal to the Fourier transform of the discrete

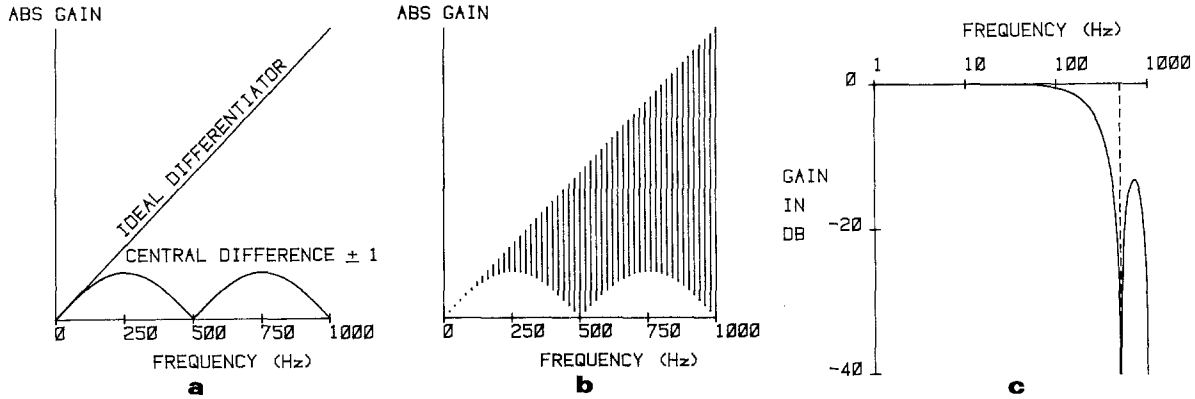


Fig. 1. **a** Absolute value of gain for a true derivative and a derivative calculated with a two-point central difference algorithm, Eq. (2) with 1000 Hz sampling rate. **b** The difference between these two curves suggests that the two-point central difference algorithm can be modeled as an ideal derivative in series with a low-pass filter. **c** A Bode diagram of the equivalent low-pass filter. There is no phase shift with a two-point central difference algorithm. To prevent aliasing artifacts, frequencies above the dashed line must be removed with analog filters

signal,  $Y(\omega T)$ , at the sampling points, and the ratio becomes

$$\left| \frac{\dot{Y}(\omega T)}{\dot{X}(\omega)} \right| = \frac{\sin \omega T}{\omega T}. \quad (4)$$

When this ratio is plotted on log-log scales, as is traditional for filters, the low-pass filter characteristics of the two-point central difference algorithm can be seen, Fig. 1c. This ratio is within 1% of unity for frequencies below 39 Hz, and is within 10% of unity for frequencies less than 125 Hz. This algorithm acts as a low-pass filter that attenuates the signal by 3 dB at 221 Hz and by 4 dB at 250 Hz.

Using  $y[(k-2)T]$  and  $y[(k+2)T]$  in (2) would yield the same result as using  $y[(k-1)T]$  and  $y[(k+1)T]$  at half the sampling rate. Doubling the spread halves the bandwidth of the algorithm from 221 Hz to 111 Hz. Using  $\pm 3$  points (a spread of 6) would produce one third the bandwidth, 74 Hz

$$\text{bandwidth (in Hz)} = \frac{0.443 \text{ sampling rate (in Hz)}}{\text{spread}}$$

If a real-time, or causal, derivative algorithm were desired, (2) could be modified

$$\dot{y}(kT) = \frac{y(kT) - y[(k-1)T]}{T}. \quad (5)$$

This backward difference algorithm has a bandwidth that is twice as large as the two-point central difference algorithm using  $y[(k+1)T]$  and  $y[(k-1)T]$ , but it is less accurate because of the digital computations involved (Young and Gregory, 1972; Marble et al., 1981), and its phase differs from that of a true derivative by  $-\omega T/2$ . We do not use such a backward difference algorithm because the phase shift precludes transferring information, such as the start or stop of a

movement, between the position and velocity records. However, for real-time applications or for latency measurements such an algorithm would be preferred (van der Tweel et al., 1980). (For latency measurements the target and eye movement data should also be filtered by identical analog low-pass filters.)

To graphically illustrate the frequency behavior of the two-point central difference differentiation algorithm we summed a large number of sinusoids to create a discrete waveform with one data point per millisecond. We applied a 512 point Fast Fourier Transform (FFT) with a Hamming window to this waveform, and computed its power spectral density. Next, we calculated the first derivative of this waveform with the 221 Hz bandwidth two-point central difference algorithm, Eq. (2) with a 1000 Hz sampling rate, and the second derivative of this waveform with a 160 Hz algorithm, Eq. (2) applied twice. Then we computed their power spectra shown in Fig. 2. This algorithm performed well below 120 Hz.

#### *Frequency Characteristics of Human Saccadic Eye Movements*

In our specific application we measured human eye movements in order to develop clinically useful diagnostic tools and to help understand how the human brain controls movement. We performed a power spectral density analysis of our data to help us select an optimal sampling rate, and optimal measurement and computational bandwidths. To save memory space the measurement and computational bandwidths should not be larger than the bandwidth of the physiological system being studied. On the other hand, information will be lost if the measurement and computational bandwidths are too small.

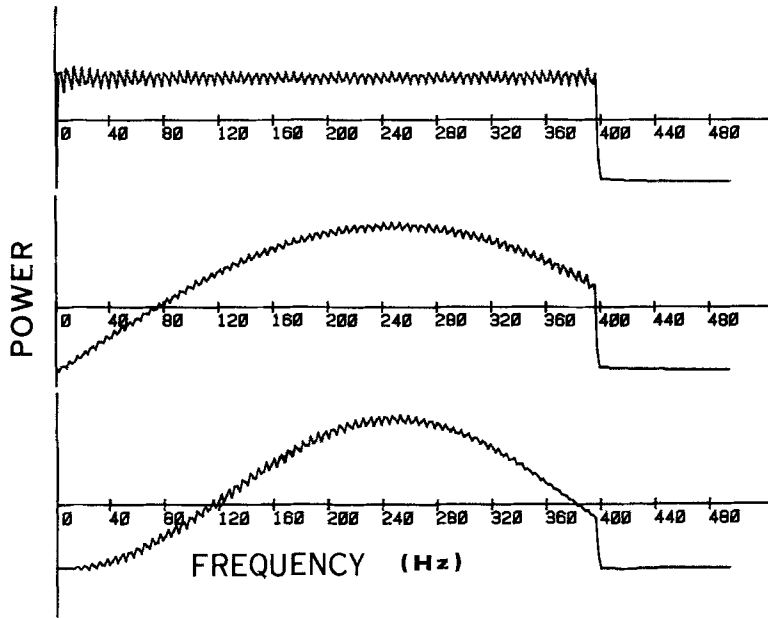


Fig. 2. Power spectral densities of (top) a waveform composed of the sum of 133 sinusoids with frequencies between 1 and 400 Hz, (middle) its first derivative, and (bottom) its second derivative. Between 1 and 400 Hz the spectrum of the original waveform should be a horizontal line; the spectrum of its first derivative should be linear; the spectrum of the second derivative should be parabolic. Departures indicate limitations of the derivative algorithm. This is a linear plot of power versus frequency

The movements of each eye were measured with a standard photoelectric system (Bahill, 1981; Bahill et al., 1981). Target and eye movements were amplified (0–300 Hz bandwidth), passed through a 12 bit analog to digital converter sampling at 1000 Hz, and were then stored on a disk in the computer. The linear range for the measurement of horizontal eye movements extended  $\pm 10$  deg from primary position. Linearity was assessed while the subject tracked a target moving sinusoidally. Calibration factors were derived while the subject tracked a target that jumped between points  $\pm 5$  deg from primary position. Calibration factors for each eye were computed by averaging one to two seconds of data from four to ten manually selected periods when the eye was stationary and looking at a target. Instrumentation noise was less than ten millivolts. It was one thousandth of the full scale range. One minute of arc movements have been recorded with this equipment. When all the limiting factors were considered it was concluded that for ten degree saccades eye position was accurate to 0.1 deg, and eye velocity was accurate to 5 deg/s (Bahill et al., 1981).

We calculated the power spectral densities of position, velocity, and acceleration records for typical eye movements and eyeblinks of normal human subjects. The bandwidths of the measurement and computation processes for these records were 300, 221, and 160 Hz respectively for position, velocity, and acceleration. Figure 3 shows these spectra for a typical 10 deg saccadic eye movement. The power in the position spectra was attenuated by 40 dB at 33 Hz. The power in the velocity spectra was attenuated by 40 dB at 40 Hz. The acceleration spectra seemed to represent only

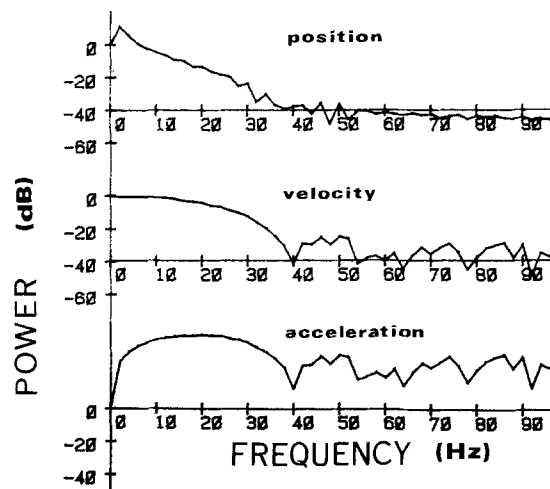


Fig. 3. Power spectral density of (from top to bottom) position, velocity, and acceleration for a typical ten degree human saccadic eye movement. This is a log-linear plot with the ordinate in decibels (dB) and abscissa in hertz (Hz)

noise above 40 Hz. We studied a variety of saccadic eye movements between 0.5 and 20 deg in magnitude: the power in the position spectra was always attenuated by 40 dB before 50 Hz; the power in the velocity spectra was always attenuated by 40 dB before 74 Hz; all acceleration spectra resembled Fig. 3C, there was no noticeable information above 45 Hz. Use of a 40 dB criterion meant that the power at higher frequencies was less than 1% of the power at low frequencies. This was a conservative guideline.

These data guided the selection of equipment and algorithms for our laboratory. We calculated the eye

velocity using

$$\dot{y}(kT) = \frac{y([k+3]T) - y([k-3]T)}{6T}. \quad (6)$$

For a 1000 Hz sampling rate this produced a velocity record with a 3 dB bandwidth of 74 Hz. We calculated eye acceleration from the eye velocity record using

$$\ddot{y}(kT) = \frac{\dot{y}([k+4]T) - \dot{y}([k-4]T)}{8T}. \quad (7)$$

A two-point central difference algorithm using  $\pm 4$  points has a bandwidth of 55 Hz. When it is used on data that have already been filtered by the velocity algorithm of (6), it produces an acceleration record with a 3 dB bandwidth of 45 Hz. A one step algorithm was tried for calculating the acceleration directly from the position, but it offered no advantages. The two-point central difference algorithm has no phase shift, so the position, velocity, and acceleration records are temporally aligned.

This frequency domain analysis indicates that maintaining a velocity channel bandwidth of 74 Hz puts the most severe restriction on the analog to digital conversion rate; 333 Hz minimum. This suggests that a 100 Hz analog low-pass filter and a 333 Hz sampling rate would be sufficient for studying position, velocity, and acceleration of human saccadic eye movements. We actually used a one kHz sampling rate for our analog to digital conversions because this rate was large enough to capture all the available information, yet low enough so that it did not swamp the storage device with superfluous data. We found that the peak velocity only lasted one or two milliseconds. Therefore, sampling at a lower rate often missed the peak and produced lower apparent peak velocities. A one kHz sampling rate also allowed duration resolution of one millisecond. This sampling rate requires an analog filter with a cutoff frequency less than 500 Hz. We used a 300 Hz analog low-pass filter because it was available and suitable.

These results are important because, in the past, various investigators, knowingly or unknowingly, used different bandwidths and got different results. If bandwidths at least as large as those outlined here would be used by all investigators there would be less interlaboratory variation in saccadic eye movement parameters.

**Note added in proof.** Using  $y([k-2]T)$  and  $y([k+2]T)$  in (2) would yield the same result as using  $y([k-1]T)$  and  $y([k+1]T)$  at half the sampling rate and twice the gain. In general

$$\dot{y}(kT) = \frac{y([k+n]T) - y([k-n]T)}{2nT},$$

where  $n$  is an integer. Increasing  $n$  decreases the bandwidth without changing the gain. The 3dB bandwidth is equal to  $0.443/nT$ .

Each eye movement record (position, velocity, or acceleration) has a different, optimal bandwidth for removing noise while retaining relevant information. The time and expense involved in filtering derivatives can be avoided by using the two-point central difference algorithm as the low-pass filter. A two-point central difference can be considered to be an ideal differentiator in series with a low-pass filter.

*Acknowledgements.* We thank Drs. Richard M. Stern and Jack D. McDonald for helpful suggestions. This research was partially supported by National Science Foundation grant ECS-8121259, National Institutes of Health grant R23 EY 02382, and by a grant from the United States Naval Air Systems Command/Naval Training Equipment Center.

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Received: November 25, 1981

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