

Frequency Limitations and Optimal Step Size for the Two-Point Central Difference Derivative Algorithm with Applications to Human Eye Movement Data

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Abstract—There are many algorithms for calculating derivatives. The two-point central difference algorithm is the simplest. Besides simplicity, the two most important characteristics of this algorithm are accuracy and frequency response. The frequency content of the data prescribes a lower limit on the sampling rate. The smoothness and accuracy of the data determine the optimal step size. We discuss the low-pass filter characteristics of this algorithm and derive the optimal step size for two types of human eye movement data. To calculate the velocity of fast (saccadic) eye movements, the algorithm should have a cutoff frequency of 74 Hz. For typical slow (smooth pursuit) eye movements, a step size of 25 or 50 ms is optimal.

INTRODUCTION

Computing derivatives is an important facet of many control and data analysis tasks. Many derivative algorithms exist; the simplest is the two-point central difference algorithm defined by the following equation:

$$\dot{y}(kT) = \frac{y([k+n]T) - y([k-n]T)}{2nT}. \quad (1)$$

In this equation T is the *sampling interval* (in seconds), nT is the *step size* ($n = 1$ for the simplest case), and k is the index for discrete time. The output of this equation can be computed with integer arithmetic and simple shift operations, which makes it appealing for microcomputer applications.

More complicated derivative algorithms exist [1]–[6]; for some data they should be preferred. For each application, the choice of an algorithm depends upon simplicity, desired accuracy, frequency characteristics of the algorithm, and characteristics of the data being analyzed. Intuitively, if more points are used for a calculation, the derivative estimate should be more accurate. However, one experimental comparison of five derivative algorithms found the two-point central difference algorithm to be the most accurate technique for 12-bit sampled data [4]. Usui and Amidror [6] investigated several integer arithmetic derivative algorithms and found the

Manuscript received May 21, 1982; revised September 27, 1982. This work was supported in part by the National Science Foundation under Grant ECS-8121259 and by the U.S. Naval Air Systems Command/Naval Training Equipment Center. A preliminary version of this note (including Figs. 2 and 3) was included in the Proceedings of the IEEE Computer Society International Conference on Medical Computer Science/Computational Medicine (MEDCOMP '82), Philadelphia, PA, September 24, 1982.

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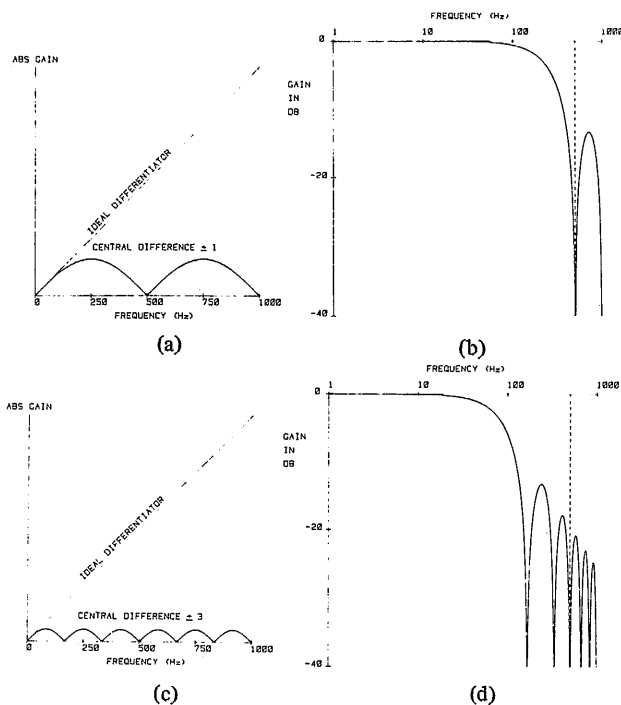


Fig. 1. (a) and (c) Absolute value of gain for a true derivative and a derivative calculated with a two-point central difference algorithm. (b) and (d) Bode diagrams of the equivalent low-pass filters. There is no phase shift with a two-point central difference algorithm. To prevent aliasing, frequencies above the dashed line must be removed with analog filters. Based on a figure from [7].

best derivative algorithm to be a two-point central difference algorithm with a built-in digital low-pass filter.

Two parameters in (1) must be selected for each application; the sampling interval T and the step size nT . We will discuss the frequency limitations of the two-point central difference algorithm, and derive the optimal step size for two types of human physiological data.

FREQUENCY LIMITATIONS

For sinusoids, the magnitude of the derivative increases linearly with frequency. For example, if $x(t) = \sin \omega t$, then its derivative is $\dot{x}(t) = \omega \cos \omega t$. A two-point central difference algorithm produces a similar output up to a certain frequency. The frequency response of the two-point central difference algorithm, as derived by [7], is

$$\frac{\dot{Y}(\omega T)}{Y(\omega T)} = \frac{j \sin \omega n T}{n T}$$

Fig. 1(a) shows the comparison of the frequency responses of the ideal derivative and the derivative estimated with the two-point central difference algorithm. Throughout this note, we assume a 1 kHz sampling rate.

The ratio of the derivative calculated with the two-point central difference algorithm to the ideal derivative is

$$\left| \frac{\dot{Y}(\omega T)}{\dot{X}(\omega)} \right| = \frac{\sin \omega n T}{\omega n T}$$

When this ratio is plotted on log-log scales [Fig. 1(b)], as is traditional for filters, the low-pass filter characteristics of the two-point central difference algorithm can be seen. This algorithm acts as a low-pass filter that attenuates the signal by 3 dB at 221 Hz. Using $n = 2$ in (1) yields the same result as using $n = 1$ at half the sampling rate. Doubling the step size halves the bandwidth from 221 to 111 Hz. Using $n = 3$ produces one-third the bandwidth (74 Hz) as shown in Fig. 1(c) and (d).

$$\text{The 3 dB bandwidth (in Hz)} = 0.443/2nT \quad (3)$$

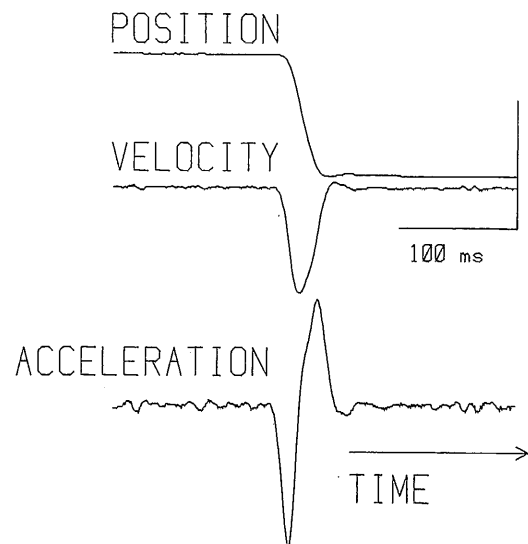


Fig. 2. Position velocity and acceleration as functions of time for a typical 10° saccadic eye movement. The calibration bar represents 10° , $500^\circ/\text{s}$, $30\,000^\circ/\text{s}^2$, and 100 ms. Position channel bandwidth was 300 Hz, velocity channel bandwidth was 74 Hz, and acceleration channel bandwidth was 45 Hz. This was a leftward movement of the right eye.

OPTIMAL STEP SIZE

There is an *optimal step size* that will produce the smallest error in any derivative estimate. This optimal step size is a function of the smoothness and accuracy of the data. For the two-point central difference algorithm, the maximum error in estimating the derivative has two components; one due to the inherent error in the numerical algorithm and the second due to applying (1) to noisy data. Following the development of Young and Gregory [1], let the derivative step size (nT) be called h . The maximum total error is given by

$$\phi = \frac{h^2 M_3}{6} + \frac{\epsilon}{h}$$

where M_3 is the maximum value of the third derivative in the interval and ϵ represents the maximum value of the noise in the data. Taking the derivative of this equation with respect to h and setting the result equal to zero, we find the optimal step size h_o is

$$h_o = \left| \frac{3\epsilon}{M_3} \right|^{1/3} \quad (4)$$

APPLICATIONS TO HUMAN EYE MOVEMENTS

We measure human eye movements to develop clinically useful diagnostic tools and to help understand how the human brain controls movement. We analyzed the frequency content and the noise in our data to help us select the optimal sampling interval and the optimal step size.

The movements of each eye were measured with a standard photoelectric system [8], [9]. Target and eye movements were amplified (0–100 Hz bandwidth), passed through a 12-bit analog-to-digital converter with a 1 ms sampling interval, and were then stored on a computer disk. The linear range extended $\pm 10^\circ$ from primary position. Linearity was obtained by adjusting the instruments while the subject tracked a target moving sinusoidally. Calibration factors for each eye were computed by averaging 1–2 s of data from four to ten manually selected periods when the eye was stationary and looking at a target $\pm 5^\circ$ from primary position.

We measured two types of eye movements—fast position correcting movements called *saccades* and slow velocity correcting movements called *smooth pursuit*. Typical examples are shown in Figs. 2 and 3. Their frequency characteristics and optimal step sizes were different.

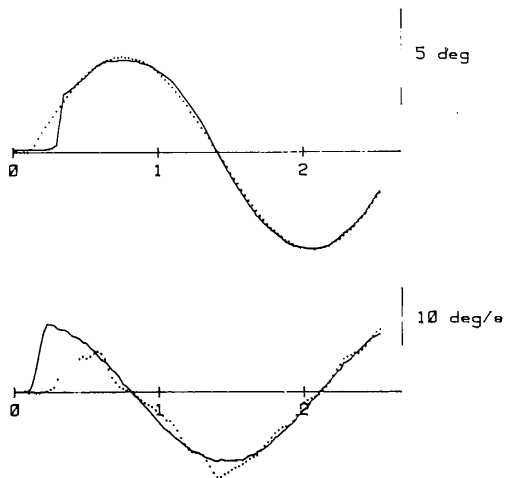


Fig. 3. Two types of eye movements—saccadic and smooth pursuit. The saccade is the fast eye movement occurring 200 ms after the start of target motion; the rest of the eye movements are smooth pursuit. The top trace shows target (dotted) and eye (solid) positions as functions of time. The bottom trace shows target (solid) and eye (dotted) velocities as functions of time. Eye position traces are solid, so that saccades can readily be seen. Target velocity traces are solid to clearly illustrate the velocity waveform that was tracked. The velocity of the saccade was so large it saturated the plotting range. The abscissa is labeled in seconds. Position and velocity bandwidths were 80 and 4.4 Hz, respectively.

SACCADIC EYE MOVEMENTS

To determine the frequency content of our data, we calculated the power spectral densities of position, velocity, and acceleration for typical eye movements of normal human subjects. The power in the position spectra fell off 40 dB before 50 Hz; the power in the velocity spectra was down 40 dB before 74 Hz [7]. Therefore, to calculate saccadic eye velocity and not filter out any significant information, the 3 dB bandwidth should be at least 74 Hz. For this bandwidth and a 1 ms sampling interval, (3) yields a step size of 3 ms. An appropriate equation is

$$\dot{y}(kT) = \frac{y([k+3]T) - y([k-3]T)}{6T} \quad (5)$$

To determine the optimal step size, we analyzed the noise in our data. When measuring 10° saccades, we set the amplifier gains so that 20° represented full scale. Instrumentation noise was then less than 10 mV, corresponding to an error of $1.2'$ of arc. Error due to the 12-bit analog to digital conversion was $0.3'$ of arc. Biological noise was about $1.5'$ of arc. In a typical data file, the maximum value of the third derivative of eye position was $6 \times 10^{16}/s^3$. (This was the maximum value of the third derivative of the signal plus noise. The third derivative of the signal alone should have been used, but it was not measurable. However, for saccades, the two values should be close in magnitude.) Using these numbers in (4) yielded an optimum step size of 3 ms.

SMOOTH PURSUIT EYE MOVEMENTS

In contrast to saccadic system data which have a large bandwidth, smooth pursuit system data have a small bandwidth. This bandwidth depends upon the frequency of the target waveform. We kept the target frequencies below 2 Hz, because humans cannot track targets moving at higher frequencies. The programs in our laboratory were optimized for the saccadic system because we studied it first. The top of Fig. 4 shows the effects of using saccadic system parameters on mixed data; the saccade is readily discernible, but the smooth pursuit velocity is buried in the noise. We had to find optimal parameters for smooth pursuit data.

Several computational processes were used on the smooth pursuit data starting with 1 ms sampling and digital low-pass

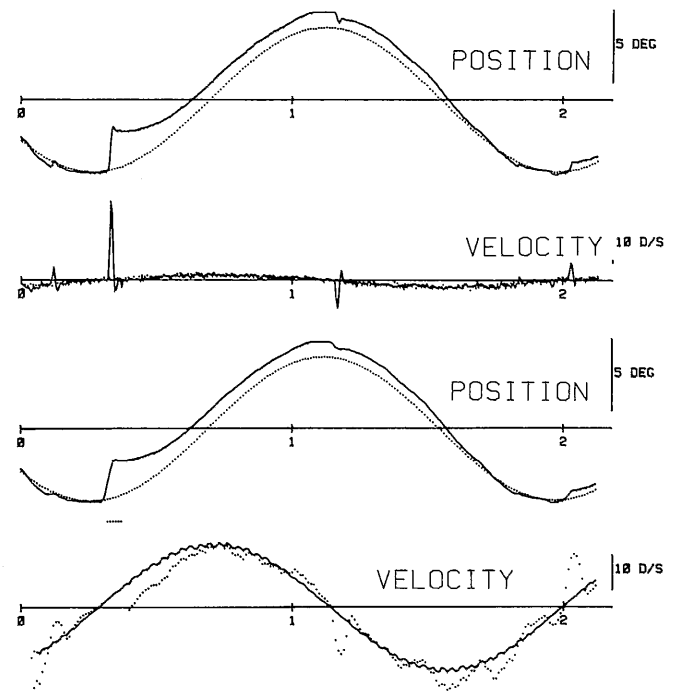


Fig. 4. Data plotted with optimal saccadic parameters (top) and optimal smooth pursuit parameters (bottom). The optimal saccadic parameters were 1 ms sampling interval, 300 Hz position channel bandwidth, 3 ms derivative step size, 74 Hz velocity channel bandwidth, and solid lines for eye velocity. The optimal smooth pursuit parameters were 5 ms sampling interval, 80 Hz position channel bandwidth, 25 ms derivative step size, 8.9 Hz velocity channel bandwidth, and solid lines for target velocity. These data show unusual off-foveal tracking—for over 1 s there is an uncorrected position error of about 1° .

filtering at 80 Hz. This filter used 35 data points representing five lobes of a sinc function in the time domain, i.e., a rectangular window in the frequency domain [9]. It took 12-bit integers as input and produced 60-bit floating point numbers as output. These floating point numbers were stored as 16-bit integers at 5 ms intervals. Only one of five points was stored, thereby producing data reduction. (The analog filter cutoff was low enough to prevent aliasing at this reduced sampling rate.) This combination of digital filtering and data reduction smoothed the data. This smoothing facilitated the computation of velocity because it reduced the quantization noise. Velocity records computed from data sampled at 1 ms intervals, filtered at 80 Hz, and stored at 5 ms intervals were smoother than records computed from data sampled at 5 ms intervals, filtered at 80 Hz, and stored at 5 ms intervals. Velocities were computed using a program that allowed the operator to interactively choose the step size. The most common choices were 25 and 50 ms—these step sizes produced bandwidths of 8.9 and 4.4 Hz, respectively.

The optimal step size for smooth pursuit data was computed. The noise in the data was the same as for saccadic eye movements, $3'$ of arc. The third derivative of the smooth pursuit data could not be used in (4) because, unlike the saccadic data, the third derivative of the signal plus noise was much greater than the third derivative of the signal alone. Therefore, the maximum value of the third derivative of the target motion was used; when the eye was tracking well, the third derivative of the eye motion would be slightly higher. For a 1 Hz, $\pm 5^\circ$, sinusoidal target motion, the maximum value of the third derivative was $1240^\circ/s^3$. Using this value in (4) yielded an optimum step size of 49 ms.

DISCUSSION

This note was written in past tense, implying that the theoretical analysis guided our choice of algorithms and parameter values. In reality, our theoretical calculations only confirmed

what years of experience had dictated. The frequency content and noise analysis arguments both indicated that the optimal step size for saccadic data was 3 ms (74 Hz bandwidth). This step size was routinely used long before these theoretical calculations were made. Robinson (in 1964) [10] recommended a bandwidth of 100 Hz for all saccadic eye movement data. In 1975, Bahill, Clark, and Stark [11] used a position channel bandwidth of 500 Hz and velocity channel bandwidths of 111 and 74 Hz. In 1981, Bahill, Brockenbrough, and Troost [9] recommended a velocity channel bandwidth of 74 Hz.

The smooth pursuit data were studied using a graphics terminal that allowed the operator to interactively choose a 3, 4, or 5 point derivative algorithm with the additional option of choosing the step size. The most common choice was the two-point central difference algorithm with a step size of either 25 or 50 ms. We used these values for over a year before we discovered that they make the 60 Hz noise fall in a notch of the filter response [e.g., one of the deep valleys in Fig. 1] and disappear from the final velocity traces. After we made our theoretical calculations, we realized that we had gradually drifted to the two-point central difference algorithm with a step size of 25 or 50 ms for our velocity computations. We seldom used 20 or 30 ms, but we were not originally aware of this prejudice. The 50 ms step size is the optimal step size derived from (4) when the maximum value of the third derivative of the data was approximated with the maximum value of the third derivative of the target waveform. The actual value of the third derivative of the data should be slightly larger, which should make the optimal step size slightly smaller. The next lower step size that also cancels out 60 Hz noise is that described by using a step size of 25 ms.

The velocity algorithm of (5) uses data ± 3 points from the center. If intervening points, such as ± 2 and ± 1 , were used, the algorithm would have better filter characteristics [5], [6]. It would not have a sharper rolloff, but once the response declined, signals would remain attenuated; the second hump shown in Fig. 1(a) would be half as large. For (5), this second hump occurs at 250 Hz. If there were noise at this frequency, the intervening points should be used. However, this would require more arithmetic operations, which would slow down the velocity computation and, depending upon the sampling rate, possibly prevent real-time implementation. It would also prevent a simple calculation of the optimal step size.

The Nyquist frequency is equal to twice the highest frequency contained in the signal being sampled. It is prescribed by the low-pass filter characteristics of the amplifier connected to the analog-to-digital converter. To prevent aliasing, it must be less than or equal to the sampling frequency.¹ A 1 kHz sampling frequency requires an analog filter with a cutoff frequency of less than 500 Hz. We adjusted the cutoff frequency of our analog low-pass filter for 100 or 300 Hz, because these values were available and suitable.

The two-point central difference algorithm can be modeled as an ideal differentiator in series with a low-pass filter. The filter does not have a sharp rolloff, but on the other hand, it also does not have a phase shift, nor does it ring even for a step input.

The selection of proper algorithms and parameter values for calculating derivatives should be guided by knowledge of the frequency limitations of the data, the noise in the data, and also by intuition and experience. The extremely simple two-

point central difference algorithm is a good algorithm. More complicated algorithms should only be used if their superior performance has been demonstrated.

ACKNOWLEDGMENT

We thank Dr. R. M. Stern and J. S. Kallman for helpful suggestions.

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¹Theoreticians consider the signal being sampled to be unchangeable; they adjust their computer programs to accommodate the signal. Thus, they say the sampling frequency must be greater than or equal to the Nyquist frequency. Experimentalists consider the computer and its programs to be immutable; they adjust their amplifiers to accommodate the computer. They say the Nyquist frequency must be less than or equal to the sampling frequency.