Chapter 1 Distance Between Two Cities

A. Terry Bahill
Systems and Industrial Engineering
University of Arizona
1622 W. Montenegro Dr.
Tucson AZ 85704-1822
terry@sie.arizona.edu

Abstract Portolan charts are detailed maps of the coastline of the Mediterranean Sea. Mariners used them to navigate in the 14th to the 16th centuries. Distances represented on these charts were amazingly accurate. This paper switches to simple 21st century Internet tools for calculating distances. It uses five techniques to calculate great circle and rhumb line distances between points on the surface of the earth. The difference in distance between a great circle route and a rhumb line route depends on the average latitude, the azimuth and the trip length, according to the following simple equation:

$$\%DifferenceDistance = -0.1*Latitude - 0.004*Azimuth + 0.14*Length$$

This paper discusses the accuracy and variability of global distance calculations. Distance calculations are accurate to within a few miles. To estimate distances between cities, the distances between their airports were used. Airport location errors were around a quarter mile. So, where distance is being traded off and accuracy of a few miles is all right, then most Internet sites should be acceptable.

Keywords: portolan chart, portolan, distance accuracy, map projection, great circle, rhumb line, Plate Carrée, airport reference point, sensitivity analysis

1.1 Introduction

Purpose The purpose of this chapter is to show the accuracy of simple Internet tools that can be used to calculate distances between cities. This accuracy depends on the precision of the coordinates of the origin and destination cities and on the route chosen whether it be a great circle or a rhumb line route. The difference in distance between great circle and rhumb line routes depends on the average latitude, the azimuth and the trip length. For short trips the most important variables are the latitude and azimuth: for long trips the most important variable is the trip length. A secondary purpose of this chapter is to exemplify portolan charts that were used by mariners in the 14th to the 16th centuries to navigate in the Mediterranean Sea and to show that for trips in the Mediterranean Sea they were quite accurate.

1.1.1 Glossary

Airport Reference Point (ARP) is defined by the FAA to be the geometric center (the centroid) of all usable runways measured to the nearest foot.

Azimuth indicates direction. It is the arc of the horizon measured clockwise from true north in degrees from 0 to 360.

Bearing has many meanings. Sometimes it is the same as azimuth. Sometimes it only has units from 0 to 90 degrees with the letters NESW providing disambiguation. Sometimes it means the direction relative to an airplane’s or a ship’s heading or movement.

A *chart* is a drawing of an area of water showing physical features such as coast lines, islands, harbors and dangers to navigation. They are used to support nautical navigation using a magnetic compass.

Distance scales are graduated bars on maps that indicate the relationship between distance on a map and the corresponding distance on the surface of the Earth.

E*quirectangular projection maps* have graticule cells that are rectangular with the same size and shape throughout the map. It is the most common type of land map projection, especially for small areas.

A grid is a network of evenly spaced horizontal and vertical lines used to identify locations on a map or another object. This contrasts with a *graticule* which is comprised of parallels of latitude and meridians of longitude for the earth.

A *great circle route* is the shortest path between two points on the surface of a sphere. An illustration of a great circle route can be produced by putting a string on a globe, anchoring the ends to the origin and destination, wiggling the string around and pulling it tight: this produces a great circle route. Great circle routes look like a straight lines on a globe and curved lines on most map projections.

A *map* is a drawing of an area of land showing physical features such as cities, rivers and coast lines. They are usually drawn on a flat surface showing how features would appear when seen from above.

Map projections are used to transform information from a three-dimensional spherical world onto a two-dimensional flat plane. Map projections almost always employ mathematical equations solved on computers.

Mercator projection maps have graticule cells that are squares at the equator and vertical rectangles closer to the poles. Lines of constant compass bearing (rhumb lines) are straight and they make the same angle with all parallels. These are the most common type of nautical charts.

Meridians are the vertical north-south lines on a globe that indicate longitude: one special example is the Prime Meridian.

Parallels are the horizontal east-west lines on a globe that indicate latitude: one special example is the Equator.

Plate Carrée projection maps are a subset of equirectangular projection maps. They use the equator as the standard parallel and as a result the graticule cells are all square.

Portolan charts are detailed maps of the coastline of the Mediterranean Sea. These charts were used by mariners in the 14^{th} to the 16^{th} centuries to navigate at sea.

A *rhumb line* route will look like a curved line on a globe and a straight line on the most common nautical chart, the Mercator projection. It has a constant bearing relative to true or magnetic north, meaning that it crosses each meridian at the same angle.

A *sensitivity analysis* is a method of finding the relative importance of variables and parameters in a model.

1.1.2 Importance of Distance Accuracy

When *game theory* is used to model competing behaviors of interacting agents, it is often crucial to correctly assess distance. For example, in military operations, the air force is a player in a two-person non-cooperative game, so its payoff depends on assessing the power, facilities and location of enemy objects. It is a game with uncertainty, where the level of uncertainty depends on the accuracy of the assessment. Similarly in business, companies are game players where their payoffs depend on the cost of transporting people and goods with airplanes and ships, their payoffs depend on distance. Accordingly, this paper assesses the accuracy and explains several techniques and simple Internet tools for computing global distances, particularly distances between two cities.

To illustrate the calculation of distances between cities, we will use these five cities in the United States of America: Minneapolis, Phoenix, Portland, San Francisco and Seattle. To increase precision, we will use the coordinates of their international airports as given in Table 1.

Table 1 Latitude and longitude of airport reference points

City	Airport code	Latitude	Longitude
Minneapolis	MSP	44.8819° N 44° 52′ 55″ N	93.2217 ° W 93° 13′ 18″ W
Phoenix	PHX	33.4342° N 33° 26′ 03″ N	112.0117° W 112° 00′ 42″ W
Portland	PDX	45.5886° N 45° 35′ 19″ N	122.5969° W 122° 35′ 49″ W
San Francisco	SFO	37.6189° N 37° 37′ 08″ N	122.3756° W 122° 22′ 32″ W
Seattle	SEA	47.4500° N 47° 27′ 00″ N	122.3117° W 122° 18′ 42″ W

<< Table 1 goes near here. >>

The data in Table 1 came from https://www.iflightplanner.com/Airports/KSEA and https://www.airnav.com/airport/KSEA. To get the coordinates of the other airports in Table 1, in the URLs above, replace SEA with MSP, PHX, PDX and SFO. For a 10,000 entry database of airport coordinates see https://openflights.org/data.html

For seaports see https://www.freightos.com/freight-resources/seaport-code-name-finder/.

This paper assesses the accuracy and variability of global distance calculations that were made using simple Internet tools to calculate distances between two cities. This accuracy depends on the precision of the coordinates of the origin and destination cities. To increase this precision, we used the coordinates of their international airports. Therefore, accuracy of calculating distances between cities depends on the accuracy of determining their airport locations. Additionally, accuracy depends on the route taken between cities, for example great circle and rhumb line routes have different distances. The difference in distance between a great circle route and a rhumb line route depends on the average latitude, the azimuth and the trip length. Also the accuracy of global distance calculations depends on the type of map projection that is used. We computed distances between airport reference points (1) by inputting airport codes or latitude and longitude coordinates into Internet applications, (2) by measuring them on maps, (3) by using 69 miles per degree of latitude and the correct miles per degree of longitude for each leg of the trip and (4) by using distance scales given on maps. We also assessed the accuracy of portolan charts that were used by mariners in the 14^{th} to 16^{th} centuries to navigate in the Mediterranean Sea: they were quite accurate. Overall, we used many techniques to assess the accuracy and variability of global distance calculations.

1.2 Accuracy of Airport Locations

In yesteryears, global distances were calculated using tables of logarithms, slide rules, maps, and charts. Now a days, GPS (the Global Positioning System) seems to be the natural choice for finding latitude and longitude coordinates. Commercial GPS receivers give the coordinates of your device and typically claim an accuracy within 15 feet horizontally. But of course, GPS receivers can be jammed, spoofed, tracked and erroneous. Many free Internet applications, such as the *GPS Coordinate Finder*, will give GPS coordinates of a building at a specified address. Many of these apps will then give a map with the address marked. This mark can subsequently be moved to any place within the building. Using several such sites we got horizontal resolution of around 20 feet. Alternatively, the *GPS Satellite* will give the coordinates of your computer. However you got them, these coordinates can then be put into apps like *PlanetCalc* to get the distance between two points. To use the GPS to calculate distances between two cities, we needed to know the exact locations of the cities. Previously we wrote that to increase precision in calculating distances between cities, we would use the coordinates of the international airports of those cities, as given in Table 1. Finding the coordinates of these big airports seemed like a simple matter. A person could merely fly to each of these airports, walk out onto the runways and measure the GPS coordinates with a hand-held device. But if security personal prevented this, then we could instead query the Internet.

When we queried the Internet, we did not get the unequivocal answers we expected. For example, for SEA (Seattle), using over a dozen Internet websites, the average latitude error was 07″ of arc with a standard deviation of 10″ and the average longitude error was 19 ±25″. This is a latitude error of 700 ±1000 feet and a longitude error of 1300 ±1700 feet, which by the Pythagorean Theorem yields a total error of 1500 feet, which is about one-quarter of a mile. Not only are these errors large, but the standard deviations are also large. Therefore, we decided that typical Internet applications are not accurate. For this reason, we used the following carefully-vetted websites:
https://www.iflightplanner.com/Airports/KSEA and https://www.airnav.com/airport/KSEA. They had the same Federal Aviation Authority (FAA) airport diagrams and gave the same crucial numbers.

These sites gave the same numbers for the latitude and longitude of the five airports listed in Table 1. But at what point were these latitude and longitude numbers for? Were they the coordinates of the air traffic control towers? No. Were they the coordinates of the rotating airport beacons? No. Were they the coordinates of the centers of the airports? No. The FAA defines the location of an airport to be the coordinates of its Airport Reference Point (ARP)(AC 150/5300-18B Appendix B). The ARP is the geometric center (the centroid) of all usable runways measured to the nearest foot. ARPs are seldom marked physically in the runway areas. These points are calculated when the airports are designed and are only changed after major runway redesign, which for big international airports is never.

The FAA knows the ARPs for all airports. However, most Internet websites that we consulted failed to give the correct coordinates. This created average distance errors of a quarter mile. This limits the accuracy that can be claimed for games, simulations and maps that use the location of airports.

When using ARPs you do not need an address and everyone can use the same precise coordinates. The runway lengths are measured to the nearest foot: the ARPs are given in degrees, minutes, and two-decimal-place seconds (one second of latitude is 100 feet). Whereas, when using the GPS you are not limited to airports: you can use your location or the address of any building. GPS receivers are accurate to 20 feet, 0.2 seconds of arc.

1.3 Great Circle and Rhumb Line Routes

On a globe, which is a three-dimensional model of the Earth, *meridians* are the vertical north-south lines indicating longitude: one special example is the Prime Meridian. *Parallels* are the horizontal east-west lines indicating latitude: one special example is the Equator.

There are two types of routes for flying or sailing between two points on the surface of the Earth; great circle routes and rhumb line routes. An illustration of a *great circle* route can be produced by putting a string on a globe, anchoring the ends to the origin and destination, wiggling the string around and pulling it tight: this produces a great circle route, which is the shortest route between two points on the surface of a sphere. Another illustration takes a hypothetical plane that passes through the two points of interest on the surface of the Earth and also through the center of the earth, the intersection of that plane and the surface of the Earth is a great circle. A great circle *route* is an arc of a great circle. In contrast, a *rhumb line* route is a curved line on the surface of the earth. It has a constant bearing relative to true or magnetic north, meaning that it crosses each meridian at the same angle. This curve is also called a loxodrome. A rhumb line route between two cities on the same meridian or parallel would follow that meridian or parallel exactly. Rhumb lines appear as straight lines on Mercator projection maps. Also far away from the poles, rhumb lines look like straight lines on equirectangular maps like that shown in figure 1. Although the arc of a great circle gives the shortest path between two points, navigating with it is difficult because the true bearing continuously changes. Following a rhumb line route is slightly longer than following a great circle route, but it is easier to navigate because it follows a constant compass direction.

Fig. 1 An equirectangular (Plate Carrée) map showing great circle (red) and rhumb line (black) routes and distances in statute miles between the airports at Minneapolis, Phoenix and Portland. A rhumb lines route between two points will lie on the equatorial side of the corresponding great circle route because great circle routes are curved away from the equator. From the *Great Circle Mapper* www.gcmap.com et al.

<< Figure 1 goes near here. >>

Figure 1 shows great circle and rhumb line routes between three cities in the United States of America. Tables 2 and 3 present the distances between these cities. Table 2 gives great circle distances that were determined with Internet applications. The data in column 3 give averages and standard deviations from over a dozen Internet applications. The data in column 4 are from https://www.airmilescalculator.com/distance/phx-to-msp/ Finally, column 5 gives data from Google Earth.

Table 2 Distances of great circle routes between airports calculated using inputs of airport codes or latitude and longitude values from Table 1

Origin and destination city and airport code		Average distance of Internet applications (statute miles ± standard deviations)	*Air Miles Calculator* distance (mi)	Google Earth distance (mi)
Minneapolis, MSP	Portland, PDX	1424 ±2.2	1426	1427
Minneapolis, MSP	Phoenix, PHX	1275 ±1.0	1276	1276
Phoenix, PHX	Portland, PDX	1009 ±0.6	1009	1009
Portland, PDX	San Francisco, SFO	550 ±0.5	550	550

<< Table 2 goes near here. >>

Next we converted the latitudes and longitudes in Table 1 into rhumb line distances using these applications www.onboardintelligence.com and https://planetcalc.com/713/ and put the results into Table 3.

Table 3 Distances and directions of rhumb line routes between city airports based on latitude and longitude values from Table 1

Origin and destination city and airport code		Rhumb line distance (mi)	Azimuth, true bearing, (degrees)
Minneapolis, MSP	Portland, PDX	1434	271.9
Minneapolis, MSP	Phoenix, PHX	1278	231.8
Phoenix, PHX	Portland, PDX	1010	326.1
Portland, PDX	San Francisco, SFO	550	178.8

<< Table 3 goes near here. >>

The rhumb line distances in Table 3 are slightly longer than the great circle distances in Table 2.

If you ask Google, "What is the distance between Denver and Washington DC?" What answers would you expect to get? The great circle distance between the two airports? The rhumb line distance between the two airports? Or the distance between the city centers? When we asked

Google, eight Internet websites gave the great circle distance between the city centers, 1491 ±2 statutes miles. Four of them gave the great circle distance between the airports, 1451 ±3 miles. None of them gave rhumb line distances. Also none of them stated what their distances were. When dealing with Internet websites, you often get what you ask for. So be careful and ask for what you actually want.

Table 4 Latitude and longitude of airports and city centers

City center and Airport code	Latitude	Longitude	Location of airport relative to city center
Washington DC	38° 54′ 02″ N	77° 02′ 13″ W	23 miles W
IAD, Dulles	38° 56′ 51″ N	77° 27′ 36″ W	
Denver CO	39° 44′ 20″ N	104° 59′ 04″ W	19 miles ENE
DEN	39° 51′ 42″ N	104° 40′ 23″ W	

<< Table 4 goes near here. >>

Denver and Washington were chosen for Tables 4 and 5 because their airports are far away from their city centers. Therefore, it was easy to see if the distances given were between the city centers, 1491 miles, or between the airports, 1451 miles. The rhumb line distances were also distinct at respectively 1499 and 1458 miles. Thus, for four different concepts, we had four different numbers that were *distinct*.

Table 5 Distances between Denver and Washington DC, city centers and airports

Average and standard deviation (miles)	Actual distance (mi)	Description
Eight web sites 1491 ±2	1493	Great circle distance between city centers
Four web sites 1451 ±3	1452	Great circle distance between airports
	1499	Rhumb line distance between city centers
	1458	Rhumb line distance between airports

<< Table 5 goes near here. >>

When flying or sailing between two points on the surface of the Earth, navigators can choose a great circle route or a rhumb line route. The great circle route will be shorter but the rhumb line route will be easier to navigate.

1.3.1 Difference in Distances Between Great Circle and Rhumb Line Routes

The great circle route gives the shortest distance between two points on the surface of the Earth. Its distance is always less than or equal to the distance of a rhumb line route. The difference in distance between a great circle route and a rhumb line route depends on the average latitude of the trip, the azimuth, and the trip length. (1) For two points on the Equator the routes are the same and hence the difference is zero. The difference increases as the average latitude increases toward the poles. (2) For points that are directly north-south of each other on a meridian (zero degrees azimuth) these two routes are also the same. However, as the bearing becomes more easterly or westerly, differences between great circle and rhumb line routes increase and then decrease. The maximum route differences due to bearing alone are on a course with an azimuth of around 60°. (3) As the trip length increases the differences increase because the great circle path is curved away from the equator and therefore intersects the meridians at higher latitudes, which have fewer miles per degree of longitude, than the rhumb line path. These differences accumulate as the route becomes longer. Therefore the differences in distance between great circle and rhumb line routes depend on average latitude, azimuth and trip length.

We needed to compare the effects of these three variables. Therefore, they had to have the same units. So, we scaled the length into degrees of arc with this equation.

$$Scaled\ Length \approx \frac{rhumb\ line\ distance}{69.17 \operatorname{Cos} \frac{\pi}{180}\left(\frac{Lat_1 + Lat_2}{2}\right)}$$

where *Scaled Length*, Lat_1 and Lat_2 are in degrees and *rhumb line distance* is in miles. The $\pi/180$ is needed for applications like Excel that expect the argument to be in radians.

In figure 2, the blue line shows the percentage difference in distance between a great circle route and a rhumb line route for an east-west trip between two points at 45° N latitude as a function of the scaled **length**. This difference is small for most trips, for example, it is less than 1.5 percent for trips less than 50 degrees in length, which includes almost all trips within the contiguous united states. But this difference really takes off for longer trips.

In the same way, the gray line shows the percentage difference in distance between a great circle route and a rhumb line route for an east-west trip that has a scaled length of 60 degrees (roughly trans-Atlantic trips) as a function of **latitude.** This difference goes from zero near the Equator to around four percent near the poles.

Finally, the orange line shows the percentage difference in distance between a great circle route and a rhumb line route as a function of **azimuth** or bearing. The scaled length of the trip and the latitude of origin were held constant at 40 and 30 degrees respectively. This difference due to azimuth starts at zero for points that are directly north and south of each other, it increases to around one percent for points at a 60 degree angle, and returns toward zero as the direction between the points becomes horizontal. This azimuth is, of course, the bearing between the two cities which is the azimuth of the rhumb line route, not the azimuth of the great circle route that constantly changes during the trip.

Figure 2 shows three particular examples of the dependence of the percentage difference between great circle and rhumb line routes on average latitude, azimuth and trip length. Example sets with different values for the variables held constant produced curves that were shifted left-right or up-down, but the general shapes remained. The data underlying figure 2 allow us to make the following generalizations.

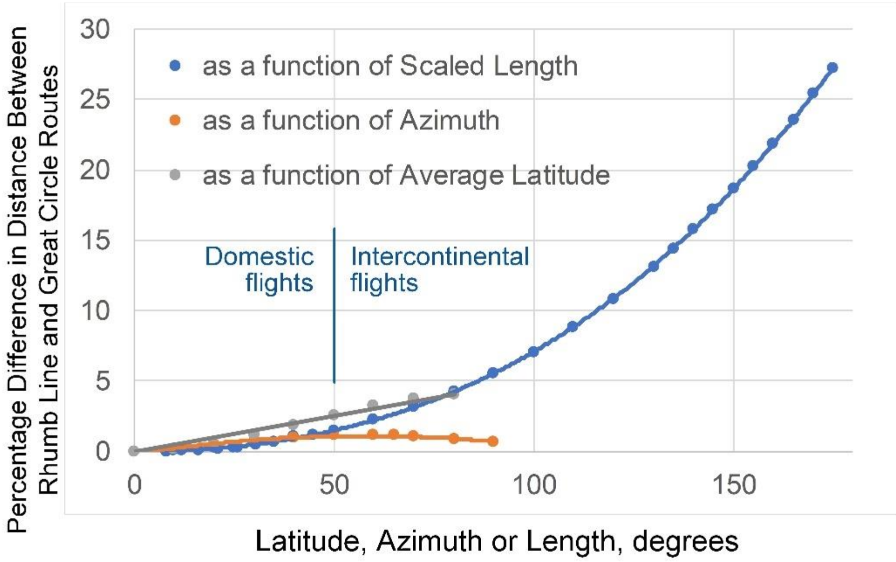

Fig. 2 Three examples of the function percentage difference in distance for great circle and rhumb line routes. Data were computed with https://planetcalc.com/713/ and https://planetcalc.com/722/

<< Figure 2 goes near here. >>

In general, the percentage difference in distance between a great circle route and a rhumb line route increases

- as the average latitude increases (almost linearly),
- as the trip length increases (almost parabolically) and
- as the azimuth increases from 0 to around 60 degrees, then it decreases as azimuth goes from 60 to 90 degrees.

These results are in concert with old navigation books (e. g. Dutton's) which have written that the difference in distance between a great circle route and a rhumb line route generally increases as the

- average latitude increases,
- difference in longitude increases and
- difference in latitude decreases.

These old navigation books simply used latitude, difference in longitude and difference in latitude as inputs whereas we used average latitude, trip length and azimuth.

These ideas about the difference in distances were quantified with https://www.movable-type.co.uk/scripts/latlong.html and confirmed with www.onboardintelligence.com. The resulting data were put into the applications https://planetcalc.com/713/ and https://planetcalc.com/722/ . These websites returned the distances for the great circle and rhumb line routes for the data of figure 2. We computed the percentage difference in distance (*%DifferenceDistance*) for these two routes for each line in the database. Then we performed a linear regression analysis with inputs of the average latitude of the trip, the azimuth, the scaled trip length and the trip length squared. The percentage difference in distance between great circle and rhumb line routes was the output. We got the following best fit equation for our data:

$$\%DifferenceDistance = 0.02*AverageLatitude + 0.01*Az - 0.08*Length + 0.001*Length^2$$

Although R^2= 0.991, this equation is not precise because it was not derived mathematically and the coefficients were rounded off. It is merely the result of a linear regression analysis of six dozen data points. The purpose in deriving this equation was to allow us to do a sensitivity analysis, which would reveal the relative importance of these three variables.

1.3.1.1 Sensitivity analysis

In order to find the relative importance of average latitude, azimuth and trip length, we needed to compute their sensitivity functions. Specifically, we wanted to use the semirelative-sensitivity of the function *F* with respect to the variable α , which is defined as

$$\tilde{S}_{\alpha}^{F} = \left.\frac{\partial F}{\partial \alpha}\right|_{\text{NOP}} \alpha_0$$

where NOP and the subscript 0 mean that all variables and parameters assume their nominal operating point values (Smith, Szidarovszky, Karnavas and Bahill, 2008).

For example, let us choose the flight from Phoenix to Minneapolis,

$$Average\ Latitude_0 = 39.16^{\circ} \approx 39^{\circ}\,\text{N}$$

$$Lat_{1_0} = 33^{\circ}\,\text{N}$$

$$Long_{1_0} = 112^{\circ}\,\text{W}$$

$$Az_0 = 51.85 \approx 52^{\circ} \text{ and}$$

$$Scaled\ Length_0(1278\text{ mi}) = 23.83^{\circ} \approx 24^{\circ}$$

The sensitivity functions become

$$\tilde{S}_{Latitude}^{\%DD} = 0.02 AverageLatitude_0 = 0.02 \times 39 \approx 0.6$$
$$\tilde{S}_{Az}^{\%DD} = 0.01 Az_0 = 0.01 \times 52 \approx 0.6$$
$$\tilde{S}_{Length}^{\%DD} = -0.08 + 2 \times 0.001 \times \left.Length\right|_{\text{NOP}} Length_0 \approx -0.1$$

These numbers may look strange. But they are correct. They look odd because of roundoff errors. The ranking of these sensitivities depends on the trip length. For US air carriers the average trip length is 1200 miles. For average trips, the most important variables are *Average Latitude* and *Azimuth*: for trips with *Length* > 52° = 2800 miles, the most important variable becomes the *Length*. Please note that this is not the point in figure 2 where the "as a function of Scaled Length" curve intercepts the "as a function of Average Latitude" curve. The reason for this is that sensitivity functions are evaluated for *small* changes around the nominal operating points. Whereas, the data in figure 2 were derived for *large* changes across the entire global range of realistic values. They are two different types of sensitivity analysis. So naturally they produce different but similar results.

Table 6 gives the nominal values, the range of realistic values and the semirelative sensitivity values that were computed analytically. The bigger the sensitivity, the more important the variable or parameter is for minimizing the difference in distance of the great circle and rhumb line routes.

Table 6 Typical values and first-order analytic sensitivities

Variables	Nominal values for a PHX to MSP trip (degrees)	Global range of realistic values (degrees)	$\tilde{S}_{\alpha}^{F} = \left.\frac{\partial F}{\partial \alpha}\right|_{\text{NOP}} \alpha_0$ semirelative sensitivity values
Average Latitude	39	0 to 80	0.6

Az	52	0 to 90	0.6
Scaled Length	24	0 to 180	-0.1

We only computed values for azimuths from 0 to 90 degrees because these values mapped exactly to values from 0 to 270, from 180 to 90 and from 180 to 270. The right column of Table 6 shows that the most important variable (the largest value), in terms of minimizing the difference in distances between great circle and rhumb line routes, is the average latitude. The second most important variable is the trip length. The least important variable is the azimuth.

The nominal operating point (NOP) also affects the sensitivities. For the trip from Portland to Minneapolis the nominal operating point is

$$Average\ Latitude_0 = 45.24^\circ \approx 45^\circ \text{ N}$$

$$Az_0 = 89^\circ \text{ and}$$

$$Scaled\ Length_0(1434\text{ mi}) = 29.27^\circ \approx 29^\circ$$

The sensitivity functions become

$$\tilde{S}^{\%DD}_{Latitude} = 0.02 Average\ Latitude_0 = 0.02 \times 45 \approx 0.7$$

$$\tilde{S}^{\%DD}_{Az} = 0.01 Az_0 = 0.01 \times 89 \approx 1$$

$$\tilde{S}^{\%DD}_{Length} = -0.08 + 2 \times 0.001 \times Length\Big|_{\text{NOP}} Length_0 \approx -0.1$$

For this nominal operating point, the most important variable is the *Azimuth*.

This sensitivity analysis has a flaw because $\tilde{S}^{\%DD}_{Length}$ is negative for both of these example nominal operating points. This is caused by the %DD linear regression equation having a negative slope for trip lengths less than 30 degrees. We know that this cannot be correct because the great circle route distance must always be less than or equal to the rhumb line distance. The cause of this flaw seems be noise in the input data. For example, for a trip between points (45° N, 0° W) and (45° N, 10° W) (after we eliminated data from two obviously incorrect websites) our six different distance calculators gave an average of 489.23 ±0.49 mi for the great circle distances and 489.17 ±0.68 mi for the rhumb line distances. Notably, the great circle distance is *greater* than the rhumb line distance. This means that the websites are wrong. Their primary equations do not work for points on the same parallel. Therefore, they use an approximation. However, because the standard deviations are large, these differences are not statistically significant. Our conclusion remains, for normal trips the most important variables in terms of minimizing the difference in distance between a great circle route and a rhumb line route are the *Average Latitude* and the *Azimuth*; for long trips the most important variable becomes the *Length.*

1.3.2 Specific Great Circle and Rhumb Line Routes

Some specific routes are unusual and interesting because they are exceptionally long or convoluted.

The commercial airline flight that has virtually the biggest difference in distances between great circle and rhumb line routes is for a flight from Boston (BOS, 42° 21′ 47″ N, 71° 0′ 23″ W) to Beijing (PEK, 40° 04′ 21″ N, 116° 35′ 51″ E). With a great circle distance of 6,737 miles and a rhumb line distance of 8,983 miles, it has a 25% difference in distance. Additionally, long-distance great circle routes that cross the equator are sometimes bizarre. Compare Buenos Aires

(EZE) to Beijing (PEK) and Buenos Aires (EZE) to Shanghai (PVG) with the Great Circle Mapper www.gcmap.com

However, these examples are outliers. Introductory classes and texts on map projections make a big deal about the difference in distance between great circle and rhumb line routes. They usually give examples that will save hours of flight time or days of sailing time. However, these examples are exceptions because the difference in distance between great circle and rhumb line routes for *intra*continental routes is less than two percent. For the examples in this paper, they were less than one percent.

Airlines sometimes *try* to fly great circle routes, but FAA control, concerned with congested traffic and restricted air spaces, ensures that they do not succeed. For example, on an airplane trip from Los Angeles to Seattle, the airplane takes off flying straight west. After an FAA determined safe distance, the pilot is allowed to turn north and initiate a great circle route, a route that is almost the same as the rhumb line route because the flight is almost straight north. Near the end of the trip, the airplane must shoot past Seattle, make a big U-turn and land flying straight south. Because of these maneuvers caused by runway orientations, the total flight distance is more than the great circle route distance.

Similarly, a ship sailing from Baltimore to Antwerp would like to spend the whole trip on a great circle route. But if it tried, it would run aground on Newfoundland . Therefore, it will start out on a rhumb line route. When it gets a certain distance away from shore, it could switch to a great circle route. But that would take it back into the area where it would have to burn expensive low-sulfur diesel fuel instead of low-cost heavy fuel oil. Therefore, the ship will continue on the rhumb line route a bit longer.

Other obstacles and rules might also prevent the use of great circle routes. Ships often avoid the North Atlantic in the spring and early summer because of numerous icebergs. During the cold war, the Soviet Union prohibited American flights through Soviet airspace. Today, restriction of airspace and sea lanes is practiced by Iran. The International Civil Aviation Organization's ETOPS rules require flight paths to ensure that in event of an engine failure an airplane will always be within a certain number of minutes of a preselected diversion airport. Additionally, commercial airlines eschew great circle routes and try to fly in the jet stream for domestic flights going east across the United States.

A great circle route is always shorter than or equal to a rhumb line route. However, for most airplane flights the differences are small, one or two percent. Airplanes and ships do not really follow great circle routes because such routes might be prevented by external restrictions and the true bearing on such a route would be continuously changing. Therefore, airplanes and ships might approximate a great circle route with a series of say 3 to 20 rhumb lines.

1.4 Distance Accuracy

To show the accuracy that should be expected in calculating distances, we used 14 websites to calculate the distance between San Francisco (SFO) and Seattle (SEA). The average distance was 678.819 miles with a standard deviation of 0.432. (Remember, great circle and rhumb line routes are the same for north-south trips like this.) The differences between websites were probably caused by assumptions about the shape and radius of the Earth, the assumed airport reference points (Table 1) and the equations used. This standard deviation is about one-half of a mile.

Normally these differences between websites would not be a problem. However, it has implications for our assumptions. The following numbers are from our best website. We assumed that SEA is directly north of SFO, actually it is off by 0.3 degrees. If SEA were directly north, then the great circle distance would be the same as the rhumb line distance: actually they are respectively 678.605 and 678.598 miles. Our second related assumption was that the Earth is a sphere, actually it is an oblate spheroid. If the Earth were a sphere, then the great circle distance between SFO and SEA would be 680.045 miles: actually it is 678.605 (using the WGS 84 model). These differences are swamped out by the differences between the 14 websites; therefore we do not have to worry about these two assumptions.

In conclusion, for games and simulations where distance is being traded off and accuracy of plus or minus a half-mile is OK, then most Internet sites should be acceptable. But if more accuracy is needed, then care must be taken in choosing the application and the assumptions.

1.5 Plate Carrée Maps

Sixty-five types of map projections are described in a common Wikipedia site. In Appendix B we describe five of the most popular ones. In the body of this paper, we only used equirectangular projections and in particular Plate Carrée projections. Perhaps unknown to the users, these are the most common map projections.

Representing a three-dimensional spherical world on a two-dimensional flat plane usually employs some type of projection process. We will now address the problem with square projections, like those on portolan charts. But first, what characterizes a square map? All graticule cells that are five degrees latitude high and five degrees longitude wide, for example, will be square no matter where they are on the map, as in figure 3. In a small area such as the Mediterranean Sea a Plate Carrée chart would be essentially the same as a square chart drawn using latitude and longitude on a flat plane with no mathematical projection. A square chart means square in terms of degrees of latitude and longitude, not square in terms of miles. The reason we used the term Plate Carrée is simply because that is the term used on the Internet for square maps of this nature. In order to use a map to measure distances between points we need to be certain of the projection that was used in making that map. In the next section, we use a Plate Carrée map projection.

1.5.1 Map Requirements

The map used in figure 3 had eight mandatory requirements: (1) it shall cover the United States of America from 25 to 50 degrees North and from 65 to 125 degrees West, (2) it shall be *Plate Carrée,* equirectangular is not good enough, (3) it shall contain lines of latitude and longitude every five degrees, in order to prove that it is square, (4) it shall contain outlines of the states, to be used to construct distance scales, (5) it shall indicate the positions of the airports of San Francisco, Portland, Phoenix and Minneapolis, which shall be labeled "San Francisco, Portland, Phoenix and Minneapolis" not as "MSP, PHX, PDX and SFO," (6) it shall be large enough (say one Mb) to allow an 8 by 10 inch figure to have 300 dpi print resolution, (7) it shall use a standard color model such as RGB and (8) it shall be in an electronic format such as jpeg or png. It was difficult for us to find or construct a map that fulfilled these requirements.

The Plate Carrée projection is an equirectangular projection in which the standard parallel is the equator. An alternative portolan chart that we considered for figure 3 was an equirectangular projection in which the standard parallel was at 35 degrees N latitude, this latitude is about the center of the Mediterranean Sea. Most maps of less than continental size that we found on the Internet that claimed to be Plate Carrée were actually equirectangular with the standard parallel centered on the area of interest.

1.5.2 Measuring Distances

Because we specified and built the map of figure 3, we are *certain* that it is a Plate Carrée map. The 5 by 5 squares on the right side of the map, superimposed on the latitude-longitude graticule, prove that this is indeed a square Plate Carrée projection. We used the dimensions of the state of Wyoming to create the distance scale. We will now use it to measure distances between Minneapolis, Phoenix and Portland.

However, first a warning: equirectangular charts, particularly square charts as in figure 3 and all other portolan charts, have a problem, namely different routes yield different distances between the origin and the destination (Gaspar, 2007).

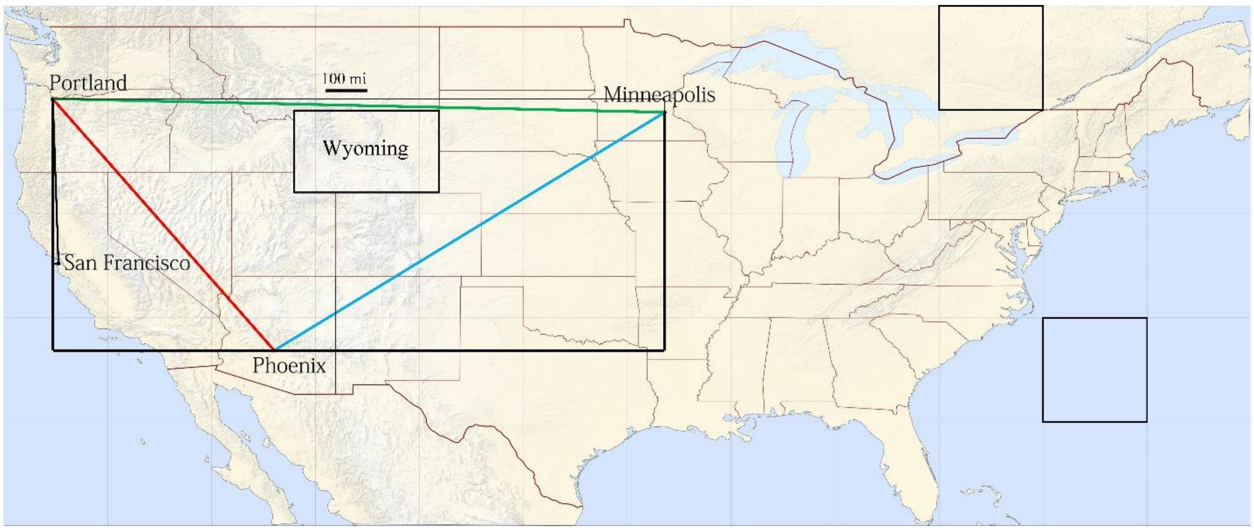

Fig. 3 A Plate Carrée map of the United States. The base map was created by Daniel Feher, Freeworldmaps.net.

<< Figure 3 goes near here. >>

Table 7 Airport to airport distance and direction measured on figure 3. Using the top of Wyoming for a distance scale yields 60.71 mi/Vcm, where Vcm stands for Visio centimeters, or "centimeters on *my* map."

Origin and destination city and airport code		Rhumb line distance and azimuth (Vcm, mi, degrees)	Vertical N-S distance (Vcm, miles)	Latitude of the horizontal leg	Horizontal E-W distance (Vcm, miles)	Total distance by Pythagorean Theorem (miles)
Minneapolis, MSP	Portland, PDX	23.82 1446 271.3°	0.52 32	45.2° N	23.81 1446	1446
Minneapolis, MSP	Phoenix, PHX	17.79 1080 238.6°	9,28 563	33.44° N	15.19 922	1080
Phoenix, PHX	Portland, PDX	13.02 790 311.4°	9.77 593	33.44° N	8.61 523	791
Portland, PDX	San Francisco, SFO	6.39 383 177.9°	6.39 388	37.62° N	0.23 14	383

<< Table 7 goes near here. >>

Except for the MSP-PDX route, which is near the distance scale, these distances are all smaller than those in Tables 2 and 3. However, the differences are orders of magnitude larger than our average measure-remeasure error on any particular map, which was 2 miles and 0.2 degrees.

1.5.3 A Trip from Minneapolis to Portland

If we were to use the map of figure 3 and the numbers in Table 7 to fly from Minneapolis to Portland, we would fly 1446 miles with a true bearing of 271 degrees and we would arrive in Portland (45° 21′ N, 122° 47′). However, if we were to fly from Minneapolis to Phoenix to Portland, then we would first fly 1080 miles bearing 239 degrees: this route would put us at (36° 43′ N, 110° 50′ W). These coordinates were produced with www.onboardintelligence.com. Then if we flew 791 miles bearing 311 degrees, we would arrive at (44° 18′ N, 122° 06′ W). This is 85 miles south and 28 miles east of Portland. An error of 89 miles.

This is the proof of our original statement that on equirectangular square maps different routes yield different distances between the origin and the destination.

1.6 Fourth Technique

Our fourth technique for computing distances between cities computes independently the vertical and horizontal components. It uses 69 miles per degree for latitude and the *Length of a Degree Calculator* for longitude. The results have been put into Table 8.

Table 8 Distances calculated using latitude and longitude from Table 1 independently

Origin and destination city and airport code		Vertical N-S distance (degrees latitude, 1 deg = 69 statute mi)	Latitude of the horizontal leg	Horizontal E-W distance (degrees longitude, miles)
Minneapolis, MSP	Portland, PDX	0.64° = 44 mi	45.24° N	29.47° = 1439 mi
Minneapolis, MSP	Phoenix, PHX	11.43° = 789 mi	33.44° N	18.87° = 1090 mi
Phoenix, PHX	Portland, PDX	12.08° = 834 mi	33.44° N	10.6° = 612 mi
Portland, PDX	San Francisco, SFO	7.97° = 550 mi	37.62° N	0.3° = 16 mi

<< Table 8 goes near here. >>

The total distance cannot be calculated with the Pythagorean Theorem because these are large spherical triangles, not small plane triangles. Otherwise, the technique does give numbers almost the same as the other rhumb line techniques for nearly horizontal or vertical routes. For example, 550 versus 550 miles for PDX to SFO and 1434 versus 1439 miles for MSP to PDX.

For these calculations we needed to know the length of one degree of latitude and the length of one degree of longitude. At 45° N, one degree of latitude is 69 miles (60 nautical miles or 111 km), for the rest of the contiguous united states this distance is within 0.3% of that value. Mariners usually say it is 69 miles everywhere on the Earth. To get the length of one degree of longitude we used this application http://www.csgnetwork.com/degreelenllavcalc.html and put the numbers into Table 9. We could just as well have used this equation:

one degree of longitude (in miles) = cosine(latitude) * length of one degree of longitude (in miles) at the equator.

As an aside, we note that at 45° N, one second of latitude is 101 feet and one second of longitude is 72 feet.

Table 9 Length of one degree of longitude from the csgnetwork and from our cosine equation

Latitude (degrees N)	Length of one degree of longitude (mi)	69.17 *cos(latitude) (mi)
49	45.47	45.38
45.35	48.69	48.61
45.2	48.82	48.74
45	48.99	48.91
44.3	49.59	49.50
37.62	54.86	54.79
37	55.31	55.24
35.15	56.62	56.56
35	56.72	56.66
33.44	57.78	57.72
33	58.07	58.01
31	59.34	59.29

<< Table 9 goes near here. >>

This technique was not useful. It only worked for routes that were primarily east-west or north-south.

There is, of course, another technique for calculating distances between cites. We could use the equations of spherical geometry. These equations exist and their accuracy is greater than the difference between a spherical Earth and an oblate Earth. However, this would merely be duplicating dozens of existing Internet websites. Therefore this technique was not pursued further.

1.7 Comparison of Results

Table 10 Comparison of the tools and results of this study

	Route	Figure 1	Table 2	Table 3	Table 6	Table 7
Tool used → Origin and destination ↓		*Great Circle Mapper*	*Air Miles Calculator*	*On Board Intelligence* and *PlanetCalc*	Plate Carrée chart of figure 4	*Length of a Degree Calculator*
Distance and azimuth from MSP to PDX (mi, degrees)	Rhumb line	1434 271.4°		1434 271.8°	1446 271.3°	1439
	Great circle	1426	1426			
Distance and azimuth from MSP to PHX (mi, degrees)	Rhumb line	1278 238.6°		1278 231.8°	1080 238.6°	
	Great circle	1276	1276			

Distance and azimuth from PHX to PDX (mi, degrees)	Rhumb line	1010 311.0°		1010 326.1°	791 311.4°	
	Great circle	1009	1009			
Distance and azimuth from PDX to SFO (mi, degrees)	Rhumb line	550 178.4		550 178.4°	383 177.9°	550
	Great circle	550	550	550		

<< Table 10 goes near here. >>

The results shown in Table 10 give us confidence in our analysis. For example, the rhumb line distances calculated between MSP and PDX with the three Internet applications are 1434, 1434, and 1439 miles. These are close. The distance calculated with the Plate Carrée chart of figure 4 is only one-half a percent larger. The great circle routes, as expected, are shorter at 1426 miles. The Internet applications also agree for the distances between MSP and PHX, PHX and PDX and PDX and SFO. The distances for the Plate Carrée map are only accurate for horizontal routes near the distance scale, at 45° N latitude of figure 4. The azimuths for the rhumb line routes in figure 1 were measured on the map and therefore their values are closer to those in figure 4 than to those in Table 3. The azimuths measured on figures 1 and 4 are only accurate for horizontal and vertical trips.

Values for the great circles routes in Table 10 are the same. Values from Internet applications for the rhumb line routes are the same, but they are different from values measured on the Plate Carrée maps.

1.7.1 Minneapolis to Portland

In summary, on a Plate Carrée chart (figure 3) the end point of our journey from Minneapolis to Portland will be different whether we use a single rhumb-line track from Minneapolis to Portland or two consecutive rhumb-line tracks: one from Minneapolis to Phoenix and the other from Phoenix to Portland. The result is that our two final positions are about 89 miles apart. This happens because it is impossible to represent the curved surface of the Earth on a computer screen or a flat piece of paper or velum without distorting distances or angles. While the distance of the north-south components of our rhumb-line tracks are preserved (meaning not distorted) the distances of the east-west components are distorted because the meridians converge near the top of the Earth. One degree of longitude at 34° N latitude is 58 miles. Whereas, one degree of longitude at 45° N latitude is only 49 miles. Therefore, Plate Carrée charts are not suitable for areas as large as the United States.

1.8 Distance Scales on Maps

Our fifth technique for computing distances between cities uses distance scales on the maps. The accuracy of this technique depends on the size and location of the area of the Earth covered by the map, the number of distance scales given, the azimuth of the true bearing and the type of map projection.

If there is a single distance scale and

If the map is of a city, then the calculation of distances should be all right.

If the map is of one of the states of the USA, and if the projection is Plate Carrée, then the calculation of distances might be off by up to 5%, else if the projection is transverse Mercator then the calculation of distances will probably be off by less than 1%.

If the map is of the Mediterranean Sea around Cyprus, Crete and the Harbor at Alexandria (an area the size of Wyoming) and the projection is Plate Carrée, then the calculation of distances will probably be off by less than 5%.

If the map is of the whole contiguous United States and if the projection is Plate Carrée, then the calculation of distances might be off by up to 32%, else if the projection is transverse Mercator then calculation of distances might be off by as much as 3%.

If there are two distance scales, one for latitude and one for longitude, and the map projection is equirectangular, then north-south measurements should be acceptable, but for areas the size of Wyoming or Colorado, east-west distances might be off by up to 6%.

If there are three distance scales, one for longitude at the top of the map, one for longitude at the bottom of the map and one for latitude, then the accuracy will depend on your measurement accuracy, the correctness of your interpolations and the accuracy of the distance scales. This should be an accurate estimate of distance.

If the map is a Plate Carrée projection of the whole Earth, don't even try to calculate distances.

1.9 Portolan Charts

Portolan charts, as shown in figure 4, are detailed maps of the coastline of the Mediterranean Sea. These charts were used by mariners in the 14th to the 16th centuries to navigate at sea. They were constructed by plotting magnetic compass directions and estimated distances between places observed by pilots at sea directly on pieces of velum with a constant scale as if the Earth were flat. The chart makers, of course, had no projection functions, that is equations that would transform coordinates on the surface of the Earth to coordinates on a flat piece of velum. Nonetheless, the resulting charts are best modeled in modern terminology as Plate Carrée charts.

Fig. 4 A portolan chart made by Jorge de Aguiar in 1492. The ‘ladder’ below the central compass rose is the distance scale. The presence of only one distance scale suggests that it is a square chart.

<< Figure 4 goes near here. >>

1.9.1 Distance Errors in the Mediterranean Sea

Now we want to discover the magnitude of the distance errors produced by using a square chart. We will use the portolan chart in figure 5, which is the southeast corner of the portolan chart made by Aguiar in 1492, figure 4. This region of the Mediterranean Sea is important because during two-thirds of these voyages sailors were not within sight of land (Gaspar, 2019) and thus they had to rely on their charts.

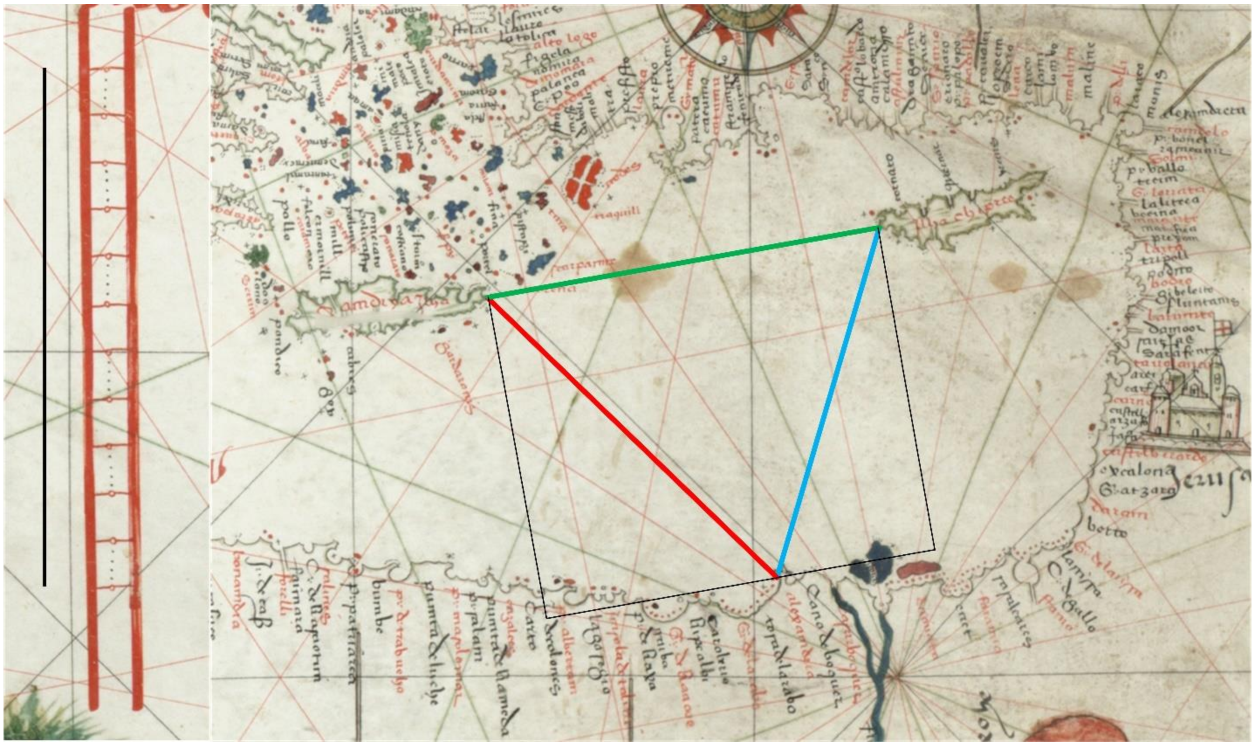

Fig. 5 A portion of Aguiar's 1492 portolan chart of figure 3

<< Figure 5 goes near here. >>

Table 11 Latitudes and longitudes for figure 5

Geographical feature	Google Earth Latitude and Longitude
North-Western tip of Cyprus	35.10° N 32.29° E
Eastern point of Crete	35.18° N 26.32° E
Harbor at Alexandria	31.19° N 29.87° E

<< Table 11 goes near here. >>

Assuming that the rungs of the 'ladder' in figures 4 and 5 are 50 miglia apart and that a miglia is 0.8 statute miles (Sheehan, 2011), then we find that on Aguiar's 1492 portolan chart, at 35° N latitude, there are 41.1 mi/Vcm, where Vcm stands for "centimeters on my map." The compass rose points toward magnetic north, which for this figure is about 11° clockwise from true north. The magnetic variation (or declination) in this region of the Mediterranean Sea in the 14^{th} and 15^{th} centuries averaged around 10 degrees.

Table 12 Distance and azimuth for routes in figure 5

Origin	Destination	Google Earth distance and azimuth (mi, degrees)	Aguiar 1492 distance and azimuth (Vcm,

			statute mi, degrees)
Western point of Cyprus	Eastern point of Crete	337.9, 272.6°	8.21, 337.7, 259.1°
Western point of Cyprus	Harbor of Alexandria	303.7, 206.7°	7.52, 309.4, 196.3°
Harbor of Alexandria	Eastern point of Crete	343.2, 324.3°	8.31, 341.6, 314°

<< Table 12 goes near here. >>

The data in Table 12 show that if we were to use Aguiar's 1492 chart of figure 5 to get from the western point of Cyprus to the eastern point of Crete we would sail 338 miles at a true bearing of 260 degrees, and we would arrive in Crete. However, if we were to sail from the western point of Cyprus to the harbor of Alexandria, followed by sailing to the eastern point of Crete, then we would first sail 309 miles at a bearing of 196 degrees: this route would put us at (31° 06′ N, 29° 51′ E). These coordinates were produced with www.onboardintelligence.com. Then if we sailed 342 miles bearing 314 degrees, we would arrive at (35° 09′ N, 26° 28′ E). This would be 10 miles south and two miles west of our target. That is not bad: it is within sight of land for a sailor in a crow's nest.

Table 13 Point to point distances calculated from the chart of figure 5

Origin and destination points		Rhumb line distance (Vcm, mi)	'Vertical' N-S distance (Vcm × 41.1 = mi)	'Horizontal' E-W distance (Vcm × 41.1 = mi)	Total distance by the Pythagorean Theorem (mi)
Western point of Cyprus	Eastern point of Crete	8.2, 338	0	8.2, 338	338
Western point of Cyprus	Harbor of Alexandria	7.5, 309	6.75, 274	3.3, 136	306
Harbor of Alexandria	Eastern point of Crete	8.3, 342	6.73, 277	4.82, 198	340
Sum of the lower two E-W legs				334	

<< Table 13 goes near here. >>

The sum of the lower two E-W legs, 334 miles, is just about the same as the direct E-W leg, 338 miles. This means that distance is not distorted in this chart of this region of the Mediterranean Sea.

The portolan chart of figure 4 has 256 rhumb line routes but no great circle routes. Great circle routes were probably not known to portolan chart makers. Great circle routes were not used by sailors until after the invention of steam power and methods of determining longitude at sea. These events occurred in the mid-19th century. Furthermore, the great circle distances between Cyprus, Crete and Alexandria are 339.76, 312.46 and 359.16 miles. Whereas, the rhumb line distances are respectively 339.82, 312.48 and 359.19 miles. The difference in these distances are tiny. Therefore, even if the portolan chart makers knew about great circle routes, great circle routes would not have been useful to them.

Until the 18th century, portolan charts were much more accurate than any other methods for showing distances between points on the surface of the Earth. Within the Mediterranean Sea their errors were on the order of ten miles. Since the year 2000, GPS has reduced these errors by an order of magnitude.

1.10 Summary

We used the following five techniques to calculate distances between points on the surface of the earth.

- We computed great circle distances between airport reference points by inputting airport codes or latitude and longitude coordinates into Internet applications.
- We computed rhumb line distances between airport reference points by inputting latitude and longitude values into a different set of Internet applications.
- We computed distances between airport reference points by measuring them on a Plate Carrée map.
- We computed distances between airport reference points by using 69 miles per degree of latitude and an Internet application to give miles per degree of longitude for each leg of the trip.
- We computed distances between airport reference points by using the distance scales given on maps.

The most accurate techniques were the first two, using validated Internet websites.

We analyzed the accuracy and variability of the locations of large airports. The FAA defines the location of an airport to be the coordinates of its Airport Reference Point (ARP), which is the geometric center of the runways. We used over a dozen Internet applications and found that most of them did not use the ARP. These websites had average errors in latitude of 07″ of arc (700 feet) and in longitude of 19″ (one-quarter of a mile). This limits the accuracy that can be claimed for games, simulations and maps that use locations of airports.

To show the accuracy that should be expected in calculating global distances, we used 14 websites to calculate the distance between San Francisco (SFO) and Seattle (SEA). The average distance was 678.819 miles with a standard deviation of about one-half of a mile. This further limits the accuracy that can be claimed for games, simulations and maps where the distance between airports is used.

Therefore, for games and simulations where distance is being traded off and accuracy of around a half-mile is all right, then most Internet sites should be acceptable. But if more accuracy is needed, then care must be taken in choosing the application and the assumptions.

If SEA were directly north of SFO, as we assumed, then the great circle distance would be the same as the rhumb line distance: actually they are 678.605 and 678.598 miles. So this assumption is fine. Next, we assumed that the Earth is a sphere, actually it is an oblate spheroid. If the Earth were a sphere, then the great circle distance between SFO and SEA would be 680.033 miles: actually it is 678.605. So this assumption is also fine. It is important to verify assumptions and we did.

The difference in distance between a great circle route and a rhumb line route depends on the average latitude of the trip, the azimuth, and the trip length. For *intra*continental-sized trips latitude and azimuth are the most important variables for minimizing this difference in distance. For *inter*continental trips the most important variable is the trip length.

Finally, we showed that distances computed on Plate Carrée charts, like portolan charts, have large errors for large areas like that of the United States. However, as long as we stay in a small region, like the Mediterranean Sea, the errors produced by using a square chart will not be large, typically just a few miles. But if we go out into the Atlantic Ocean, we should use different scales at the top and bottom of the chart and a different scale for latitude.

1.11 Acknowledgements

We thank Bruce Gissing, Ferenc Szidarovszky and Nick Bahill for valuable comment on the manuscript.

1.12 References

Bluemle JP (2002) North Dakota's Boundaries, North Dakota Geological Survey,

https://www.dmr.nd.gov/ndgs/ndnotes/boundaries/boundaries.asp

Accessed June 2019.

Gaspar JA (2007) The myth of the square chart. *e-Perimetron* 2(2):66-79.

Gaspar JA (2019) The Liber de existencia riveriarum (c. 1200) and the birth of nautical cartography. *Image Mundi* 71(1):1-21. DOI: 10.1080/03085694.2019.1529898

Sheehan KE (2014) The Functions of Portolan Maps: An evaluation of the utility of manuscript nautical cartography from the thirteenth through sixteenth centuries,, Durham University. Available at Durham E-Theses Online: http://etheses.dur.ac.uk/10645/

Smith ED, Szidarovszky F, Karnavas WJ, Bahill AT (2008) Sensitivity analysis, a powerful system validation technique. The Open Cybernetics and Systemics Journal. http://www.bentham.org/open/tocsj/openaccess2.htm, 2: 39-56, doi: 10.2174/1874110X00802010039

1.13 Appendix A. Comments about Projections

A critic who was chastising Pablo Picasso for painting Cubist abstractions pulled a photograph of his wife out of his wallet, showed it to the artist and challenged him with, “Why can’t you paint realistically, like this?”

Picasso asked, “Is that what your wife really looks like?”

“Yes,” the man replied.

“Well, she’s very small and quite flat,” retorted Picasso.

1.13.1 Assumptions Used in this Paper

1. The difference in elevation (feet above sea level) between the origin and destination has no effect on the distances calculated in this paper.
2. The altitude at which airplanes fly can be neglected.
3. Trips are made on the surface of the Earth: tunneling is not allowed.
4. This paper was written from the perspective of a person in the northern hemisphere. None of our routes crossed the equator.

5. The chart of Aguiar 1492 can be modeled as a Plate Carrée chart.
6. The Earth is an oblate spheroid following the WGS 84 model.
7. Distances are given in statute miles, unless otherwise stated.

1.13.2 Boundaries of Some Western States

At 45° N, one degree of latitude is 69 miles (60 nautical miles or 111 km), for the rest of the contiguous united states it is within 0.3% of that value. Mariners usually say that one degree of latitude is 69 miles everywhere.

Table 14 Boundaries of some western states of the USA

	Montana	Wyoming	Colorado	Arizona
Latitude of northern border	49° N	45° N	41° N	37° N
Latitude of southern border	45° N On the horizontal eastern portion	41° N	37° N	31° 20′ On the horizontal eastern portion.
Longitude of eastern border	104° 02′ 50″ W	104° 03′ 09″ W	102° 03′ W	109° 03′ W
Longitude of western border	116° 03′ W On the vertical northern portion	111° 03′ W	109° 03′ W	114° 03′ W On the vertical northern portion
Size, latitude by longitude (degrees)	4 by 12	4 by 7	4 by 7	5.7 by 5
Width along northern border (mi)	546	343	366	332
Width along southern border (mi)		366	387	
Height (mi)	276	276	276	393

<< Table 14 goes near here. >>

The strange 03′ of arc in the longitude of the defining boundaries was caused by 19th century lawmakers not using Greenwich as zero longitude. Instead they based longitude on the center of the dome of the Naval Observatory in Washington, DC, which is 77° 03' 2.3" west of Greenwich (Bluemle, 2002). Another strange boundary is the southern border of Arizona that was set by the Gadsden Purchase.

None of the states in the USA are square. Wyoming and Colorado come the closest. In terms of degrees of latitude and longitude, they are identical in size and shape on a Plate Carrée map. However, in terms of miles, they are *not* identical in size and shape. Consequently, they are not identical in size and shape on the surface of the Earth.

1.13.3 Slang Terms often Thrown Around on the Internet

As the crow flies meaning a great circle route

Air miles often meaning nautical miles (nm)

Land miles meaning statute miles (mi)

Straight line distance often mistakenly meaning great circle distance!

Plate Carrée mistakenly meaning any equirectangular projection

WGS 84 mistakenly meaning a Plate Carrée projection

These terms were presented so that if you see them, you will know that you must use caution when reading that document.

1.14 Appendix B. Guide to Common Map Projections

A common Wikipedia site gives examples of sixty-five different map projections. Most of them would only be loved by their mothers. In this appendix we describe five of the most popular ones. The Mercator projection is the most common nautical chart projection. Equirectangular and Plate Carrée are the most common land map projections. *Plate Carrée* is French for *flat square*. It is a special case of equirectangular with the equator as the standard parallel. Hopefully this appendix will help you to understand the type of projection that you are using and the consequences of using it.

Conformal projections preserve the shape of small objects. Conformal means it preserves *form*. Angles are also preserved meaning that lines of constant azimuth (rhumb lines) are straight and they make the same angle with all parallels. It also means that angles measured on the map are the same as those measured on the Earth. Meridians and parallels intersect at right angles. At any point the distance scale is same in every direction. The size of objects is distorted. The Mercator projection is an example of a conformal projection.

Equal-area projections preserve the area of objects. To do this, the properties of shape, angle, scale, or any combination of these are distorted. No flat map can be both equal-area and conformal. The US National Atlas uses a Lambert Azimuthal Equal-area projection. Two other common equal-area projections are the Albers Equal-area Conic and the Lambert Conformal Conic.

The properties of shape, area, distance and direction are mutually exclusive. A map projection that preserves one, will distort the other three.

Equidistant projections preserve distances but only from the center of the projection or along a special set of lines. For example, an Azimuthal Equidistant map centered at Minneapolis shows the correct distance between Minneapolis and any other point on the projection. It shows the correct distance between Minneapolis and Phoenix, and Minneapolis and Portland. But it does not show the correct distance between Phoenix and Portland. No flat map can be both equidistant and equal area.

True-direction projections preserve direction. They maintain some great-circle arcs, giving the directions or azimuths of all points on the map correctly with respect to the center. Unfortunately, much like the equidistant projections, this only works for one point on the map.

For the projections described in Table 15, the meridians and parallels are perpendicular. For curved lines, perpendicular means that the tangent of the curve at the intersection is perpendicular the other line.

A grid is a network of evenly spaced horizontal and vertical straight lines used to identify locations on a map or another object. This contrasts with a *graticule* which is comprised of parallels of latitude and meridians of longitude for the earth.

Table 15 Properties of common map projections. In this table approx is an abbreviation for approximately. Rows marked with an asterisk (*) would be most helpful for visually distinguishing different projections.

Name of Projection →	Mercator cylindrical	Equirectangular	*Plate Carrée*, a special case of equirectangular	Albers Equal Area Conic	Lambert Conformal Conic
*Figures in this paper	6	1, 6 and 7	1, 3, 4, 5 and 6	6	6
Type of projection	Cylindrical	Cylindrical	Cylindrical	Conic, secant form	Conic, secant form
*Lines of longitude, meridians, are	Vertical, equally spaced, parallel, straight lines	Vertical, equally spaced, parallel, straight lines	Vertical, equally spaced, parallel, straight lines	Equally spaced radii of concentric circles that converge toward the polar axis	Equally spaced radii of concentric circles that converge at the pole.
*Lines of latitude, parallels, are	Horizontal, not equally spaced, parallel, straight lines	Horizontal, equally spaced, parallel, straight lines	Horizontal, equally spaced, parallel, straight lines	Unequally spaced arcs of concentric circles, whose spacing depends on selection of standard parallels	Unequally spaced arcs of concentric circles, whose spacing increases toward the poles.
Standard parallel	Equator	Varies	Equator	Uses two	Uses one or two
Shape of 5 by 5 degree graticule cell at the equator	Square	Rectangular throughout the map	Square throughout the map	Approx Trapezoid	Approx Trapezoid
*Shape of 5 by 5 degree graticule cell near 50 degrees N latitude	Vertical rectangle	Rectangular	Square	Approx Trapezoid	Approx Trapezoid
Shape of 5 by 5 degree graticule cell at 85 degrees N	Nonexistent. This projection is useless above 80° N	Rectangular	Square	Approx Trapezoid	Approx Trapezoid
Going from the standard parallels toward the north pole *shapes of objects* are	Stretched in both directions. Areas inflate	Stretched horizontally	Stretched horizontally	Remain constant	Stretched vertically

*Going from the standard parallels toward the north pole *graticule cells*	Get taller	Remain the same	Remain the same	Get narrower	Get narrower
Shape preserving (conformal)	Yes	No	No	No	Yes
Area preserving	No	No	No	Yes	No
At every point is east/west distance scale same as north/south scale?	Yes, to about 20° north and south latitude	No	Only for small regions near the standard parallel	No	Yes, between the standard parallels
A great circle route appears as	Approx arc of a circle (except meridians and equator)	A curve	A curve	A curve	Approx straight line
A rhumb line appears as	A straight line	A straight line near the Equator, a curve above 40° N	A straight line near the Equator	An arc of a circle	A curve
*Western USA-Canadian border is a straight horizontal line	Yes	Yes	Yes	No, it is the arc of a circle following the 49th parallel	No, it is the arc of a circle following the 49th parallel
Applications	Microsoft Bing Maps, Virtual Earth, Yahoo Maps, Previous Google	This is probably the most commonly used projection	Portolan charts, ArcGIS, road maps, city maps	When area is important, such as in showing population density	Widely used. Popular with airplane pilots, USGS

<< Table 15 goes near here. >>

What is the difference between a Mercator and an equirectangular projection? In the Mercator projection *graticule cells* are squares at the equator and vertical rectangles closer to the poles. The Mercator projection is shape preserving (conformal). Lines of constant magnetic compass bearing (rhumb lines) are straight and they make the same angle with all parallels. Size is greatly distorted near the poles. In contrast, in the equirectangular projection graticule cells are the same shape everywhere and objects are stretched horizontally in the top of the map.

What is the difference between a equirectangular and Plate Carrée projection? The equirectangular projection converts the globe into a Cartesian graticule. Each rectangular *graticule cell* has the same size and shape. All the graticule line intersections are at 90 degrees. The standard parallel may be any line, but the Plate Carrée projection uses the equator. When the equator is used, the graticule cells are perfect squares, but if any other parallel is used, the graticule cells are rectangular. Most maps, of less than continental size, that we found on the

Internet that claimed to be Plate Carrée, were actually equirectangular with the standard parallel centered on the area of interest. Most global maps that we found on the Internet that claimed to be equirectangular were actually Plate Carrée.

Some projections are designed for a specific point. Then both distance and direction are accurate from this central point. Some projections are designed for a specific region. Two standard parallels might be used. Distortion is minimized in the region between the standard parallels. Unless these points or regions are specified it is dangerous to use these maps for anything except what the designer intended.

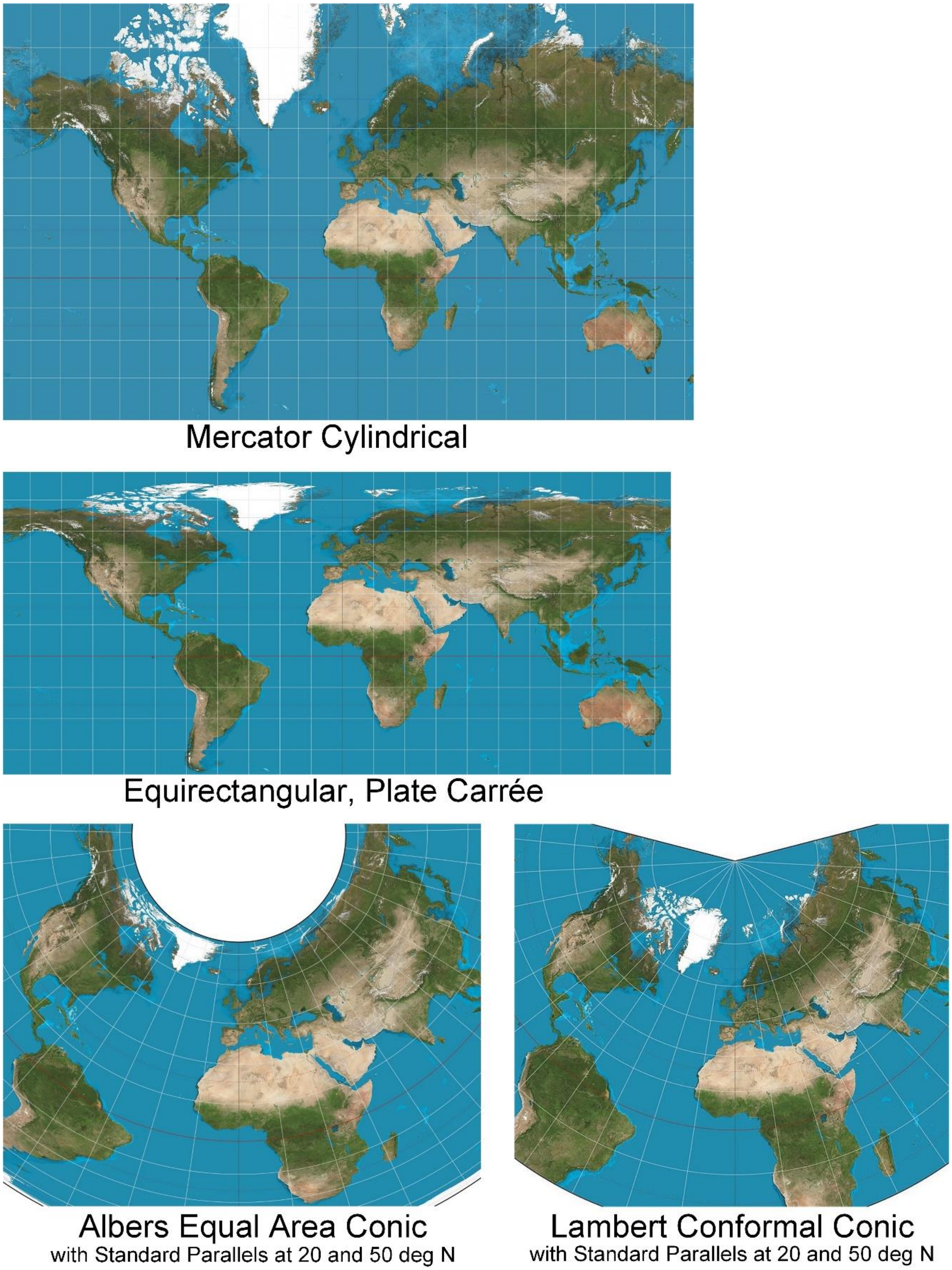

Fig. 6 The most common map projections. By Daniel Strebe - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=16115144 https://en.wikipedia.org/wiki/List_of_map_projections

<< Figure 6 goes near here. >>

If your mapmaker has not told you what type of projection was used to make the map that you are using, then you will have great difficulty drawing conclusions. If he or she has told you what projection was used, but not what standard parallels were used, then you may be equally flummoxed.

1.15 Appendix C. Second Example in the Mediterranean Sea

In the text we used Aguiar’s portolan chart of 1495 to show the magnitude of error that would be produced by using a square portolan chart in the Mediterranean Sea. We will now present a second example of this using portolan chart of Seale. We will use the portolan chart in figure 7, which is the southeast corner of the portolan chart made by Richard William Seale in 1745. This region of the Mediterranean Sea is important because during two-thirds of these voyages sailors were not within sight of land (Gaspar, 2019) and thus they had to rely on their charts.

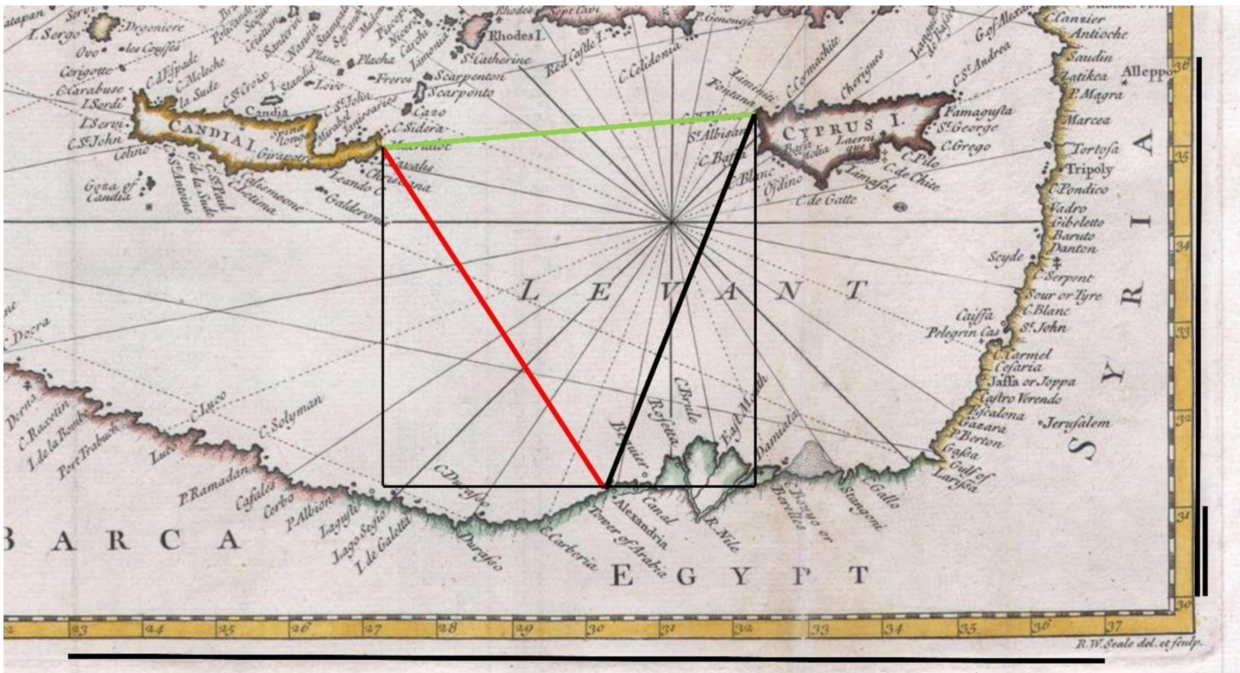

Fig. 7 “*A Correct Chart of the Mediterranean Sea From the Straits of Gibraltar to the Levant,”* by Richard William Seale, 1745, lower-right corner only

<< Figure 7 goes near here. >>

Table 16 Latitudes and longitudes for figure 7

Geographical feature	Google Earth Latitude and Longitude	Seale 1745 Latitude and Longitude
North-Western tip of Cyprus	35.1° N 32.28° E	35.3° N 32.25° E
Eastern point of Crete	35.18° N 26.32° E	35.17° N 27.25° E
Harbor at Alexandria	31.2° N 29.9° E	31.2° N 30.2° E

<< Table 16 goes near here. >>

On Seal′s chart, at 35° N latitude, one degree latitude equals 68.7 statute mi ⇒ 36.5 mi/Vcm and one degree of longitude equals 56.72 mi ⇒ 36.5 mi/Vcm. Therefore, Seal's chart is square in terms of miles, but not in terms of degrees. Accordingly it is not Plate Carrée. It is probably best approximated as equirectangular with a standard parallel at 35° N latitude. He has different scales for horizontal longitude and vertical latitude. His Crete is about 53 miles too far east and his north-western point of Cyprus is about 14 miles too far north. His magnetic variation is about six degrees.

Table 17 Distance and azimuth for routes in figure 7

Origin	Destination	Google Earth distance and azimuth (mi, degrees)	Seale 1745 distance and azimuth (mi, degrees)	East-west component on Seale's chart (mi)
Western point of Cyprus	Eastern point of Crete	338, 270.7°	285.43, 265°	285
Western point of Cyprus	Harbor of Alexandria	303.42, 207.9°	308.32, 202°	114
Harbor of Alexandria	Eastern point of Crete	344.27, 325.3°	310.5, 327°	170

<< Table 17 goes near here. >>

Table 17 indicates that if we were to use Seale's chart of figure 7 to sail from the western point of Cyprus to the eastern point of Crete, we could sail 285 miles at a true bearing of 265 degrees and we would arrive in Crete. However, if we were to sail from the western point of Cyprus to the harbor of Alexandria, to the eastern point of Crete, then we would first sail 308 miles at a true bearing of 202 degrees: this route would put us at (31° 09′ N, 30° 15′ E). These coordinates were produced with www.onboardintelligence.com

Then if we sailed 311 miles bearing 327 degrees, we would arrive at (34° 56′ N, 27° 20′ E). This would be 17 miles off target. That is not bad. It is within sight of land.

Table 18 Point to point distances calculated from the chart of figure 7

Origin and destination points		Rhumb line distance (Vcm, mi)	Vertical N-S distance (Vcm × 36.5 = mi)	Latitude of horizontal leg in figure 7	Horizontal E-W distance (Vcm × 36.5 = mi)	Total distance by the Pythagorean Theorem (mi)
Western point of Cyprus	Eastern point of Crete	7.82 = 285	0.69 = 25	35° N	285	286
Western point of Cyprus	Harbor of Alexandria	8.45 = 308	7.8 = 285	31° N	3.125 = 114	307
Harbor of Alexandria	Eastern point of Crete	8.51 = 311	7.1 = 259	31° N	4.65 = 170	310

Sum of the lower two E-W legs					284	

<< Table 18 goes near here. >>

From Table 18, we see that the sum of the lower two E-W legs, 284 miles, is just about the same as the direct E-W leg, 286 miles. [This is smaller than the result we derived for Aguiar’s chart because Seale’s Crete is 53 miles too far east and his north-western point of Cyprus is 14 miles too far north.] This means that distance is not distorted on the chart for this region of the Mediterranean Sea.

Therefore, as long as we stay in a small region, like the Mediterranean Sea, the errors produced by using a square chart will not be large, typically just a few miles. But if we go out into the Atlantic Ocean, we should use different scales at the top and bottom of the chart and a different scale for latitude.

1.16 Appendix D. Study of Gaspar

This chapter was inspired by Gaspar (2007) who made a similar argument for ocean travel between Lisbon, Madeira and Terceira in the Azores.

Table 19 Latitudes and Longitudes for three airports in the eastern Atlantic Ocean

Region, Island	City	Airport	Latitude	Longitude
Portugal	Lisbon	Lisbon, LPPT	38.7813° N 38° 46′ 53″N	9.1359° W 9° 08′ 09″W
Madeira	Funchal	Madeira, LPMA	32.7975° N 32° 41′ 51″N	16.7745° W 16° 46′ 28″W
Azores, Terceira	Angra de Heroismo	Lajes, LPLA	38.7618° N 38° 45′ 43″N	27.0908° W 27° 05′ 27″W

<< Table 19 goes near here. >>

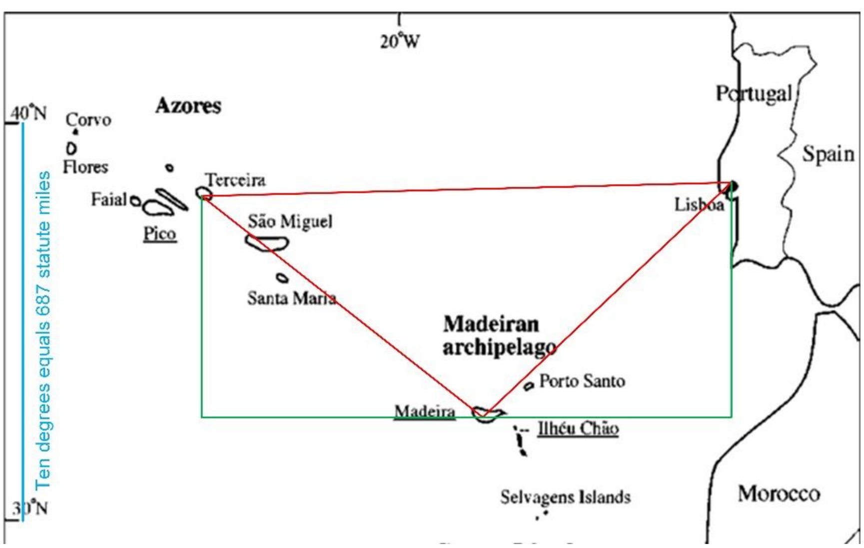

Fig. 8 The study of Gaspar (2007)

<< Figure 8 goes near here. >>

Rhumb line distances based on the latitudes and longitudes given in Table 20 were computed using http://www.pilotnav.com/airport/

Table 20 Rhumb line distances and directions based on latitude and longitude

Origin and destination city and airport code		Rhumb line distance (mi)	Azimuth, bearing (degrees)
Lisbon, LPPT	Terceira, LPLA	970	270
Lisbon, LPPT	Madeira, LPMA	600	226
Madeira, LPMA	Terceira, LPLA	715	306

<< Table 20 goes near here. >>

1.17 Appendix E. Flat Earth Equations

If the Earth were flat or if we were only interested in areas the size a state or smaller (i. e. less than 300 miles square), then the following equations could be used.

Our system could be run with two sets of inputs. In one version the inputs were (Lat_1, $Long_1$) and (Lat_2, $Long_2$) in degrees. These were used to compute the average latitude, the azimuth of the destination with respect to the origin (*Az*) and the scaled trip length (*Scaled Length*) all in degrees. In a second version the inputs were (Lat_1, $Long_1$), azimuth (*Az*) and the scaled trip length (*Scaled Length*).

The relations between the latitudes and longitudes of the origin and destination are

$$Lat_2 \approx Lat_1 + ScaledLength * \text{Cos}\, Az$$

$$Long_2 \approx Long_1 + ScaledLength * \text{Sin}\, Az$$

where Lat_1 and $Lat_2 < 90$ degrees and $Long_1$ and $Long_2 < 180$ degrees.

For a reasonability check we used the same planar model and compared the results of the following equation to the rhumb line distances.

$$Trip\ Length \approx 69.2\sqrt{(Lat_2 - Lat_1)^2 + \left[(Long_2 - Long_1)\cos((Lat_2 + Lat_1)/2)\right]^2}$$

where Lat_1, Lat_2, $Long_1$ and $Long_2$ are in degrees and *Trip Length* is in miles.

We needed a way to compare the effects of average latitude, azimuth and trip length. They had to have the same units. So, we scaled the length with this equation.

$$Scaled\ Length \approx \sqrt{(Lat_2 - Lat_1)^2 + \left[(Long_2 - Long_1)\cos((Lat_2 + Lat_1)/2)\right]^2}$$

where Lat_1, Lat_2, $Long_1$, $Long_2$ and *Scaled Length* are all in degrees.

However, the following equation was more accurate.

$$Scaled\ Length \approx \frac{rhumb\ line\ distance}{69.17\,\text{Cos}\,\frac{\pi}{180}\left(\frac{Lat_1 + Lat_2}{2}\right)}$$

where *Scaled Length*, Lat_1 and Lat_2 are in degrees and *rhumb line distance* is in miles. The azimuth can be computed with

$$Az \approx Arc\,\mathrm{Tan}\,\frac{\left|Long_1 - Long_2\right| * \mathrm{Cos}((Lat_2 + Lat_1)/2)}{\left|Lat_2 - Lat_1\right|}$$